Department of Computer Science
Vrije Universiteit Amsterdam

Midterm Exam for the course *Logic and Sets*
February 28, 2017, 13:30–15:15

This exam has 3 pages and 6 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. **Semantic entailment** (7 + 7 points)
   Investigate whether the following two statements regarding semantic entailment hold. For each statement: If it holds, argue that this is the case; if not, provide a counterexample.

   (a) If $\phi \vDash \psi$, then $\phi \vDash \psi \wedge \chi$.

   (b) If $\phi \vDash \psi$, then $\phi \vDash \psi \vee \chi$.

2. **Island puzzle** (15 points)
   On the island of liars and truth speakers everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth).

   You meet three islanders A, B and C.
   A says: "B or C is a liar, but not both."
   B says: "A is a truth speaker."
   C says: "B is a truth speaker."

   You need to determine, *by means of logical reasoning using propositional formulas*, which of these three islanders speak the truth and which ones lie. Explain your argumentation. Also check explicitly that the claims by the three islanders are in line with your conclusion whether they are truth speakers or liars.

3. **Conjunctive normal form (CNF)** (7 + 7 + 2 points)
   Transform the propositional formula $p \vee \neg(q \vee \neg r)$ to conjunctive normal form (CNF) using two different techniques. In both cases describe all the transformation steps.

   (a) Rewrite $p \vee \neg(q \vee \neg r)$ to CNF using the algorithm `CNF`.

   (b) Transform $p \vee \neg(q \vee \neg r)$ to CNF on the basis of its truth table.

   (c) Finally, state whether $p \vee \neg(q \vee \neg r)$ is a tautology, a contradiction or contingent. Explain your answer.

4. Sets (6 + 3 + 6 points)

(a) Consider three sets A, B and C in a universe U of 33 elements. Suppose we know in addition that

$$\#A = 21 \qquad \#B = 13 \qquad \#C = 15$$
$$\#(B \cap C) = 6 \qquad \#(C \cap A) = 11 \qquad \#(A \cap B \cap C) = 4$$
$$\#(A \cup B) = 26.$$

Draw a Venn diagram of the sets A, B and C, and determine the number of elements in every region of your Venn diagram.

(b) Use your Venn diagram from part (a) to determine the following numbers:

$$\#(B \cup C)', \qquad \#(C \setminus B), \qquad \#(A \cap B).$$

(c) Derive the following equality (for all sets A, B and C), using the algebra of sets:

$$(B \setminus C) \cup (B \cap A) = B \setminus (C \setminus A).$$

5. Relations (4 + 6 + 6 points)

Consider the relation R in the set $V := \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ that has the matrix representation

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

where the first row and column correspond to element `a`, the second row and column to element `b`, the third row and column to `c`, and the last row and column to `d`.

(a) Explicitly list all the elements of R in the curly-bracket notation, and draw the directed graph representation of R.

(b) Is the relation R reflexive? transitive? symmetric? anti-symmetric? Briefly explain your answers.

(c) Determine the relation $R^{-1} \circ R$, explicitly list all its elements in the curly-bracket notation, and draw its directed graph representation.

6. Ordering relations (8 + 3 + 3 points)
Let A be the subset of $\{0,1,2\}^2$ defined by

$$A := \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \langle 1,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle\}.$$

In this task, we consider the Cartesian ordering relation on A induced by the usual partial order $\leq$ on the elements of the set $\{0,1,2\}$.

(a) Use the algorithm that you have learned to construct the Hasse diagram that represents the Cartesian ordering relation on A. Explicitly write down the sets G_x and H_x obtained in the construction, and draw the Hasse diagram.

Remark: to simplify the notation, you may write 00 instead of $\langle 0,0\rangle$, 12 instead of $\langle 1,2\rangle$, et cetera.

(b) Does the set A have a smallest element according to the Cartesian ordering relation? If so, please give this smallest element. If not, please list all the minimal elements of the set A.

(c) Does the set A have a largest element according to the Cartesian ordering relation? If so, please give this largest element. If not, please list all the maximal elements of the set A.