Department of Computer Science
Vrije Universiteit Amsterdam

Midterm Exam for the course *Logic and Sets*
February 28, 2017, 13:30–15:15

This exam has 5 pages and 6 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. **Semantic entailment** (7 + 7 points)
Investigate whether the following two statements regarding semantic entailment hold. For each statement: If it holds, argue that this is the case; if not, provide a counterexample.

   (a) If $\phi \vDash \psi$, then $\phi \vDash \psi \wedge \chi$.

   (b) If $\phi \vDash \psi$, then $\phi \vDash \psi \vee \chi$.

   Solution:

   (a) This claim does not hold. A counterexample is a valuation for which ϕ and ψ are *true* while χ is *false*, because then for this valuation also $\psi \wedge \chi$ is *false*.

   (b) This claim holds. Consider a truth table for ϕ, ψ and $\psi \vee \chi$. The semantic entailment $\phi \vDash \psi$ means that in each line of the truth table, if ϕ is *true* then ψ is *true*. Clearly if a valuation makes ψ *true*, then it also makes $\psi \vee \chi$ *true*. Hence, in each line of the truth table, if ϕ is *true* then $\psi \vee \chi$ is *true*. So ϕ semantically entails $\psi \vee \chi$.

2. **Island puzzle** (15 points)
On the island of liars and truth speakers everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth).

   You meet three islanders A, B and C.
   A says: "B or C is a liar, but not both."
   B says: "A is a truth speaker."
   C says: "B is a truth speaker."

   You need to determine, *by means of logical reasoning using propositional formulas*, which of these three islanders speak the truth and which ones lie. Explain your argumentation. Also check explicitly that the claims by the three islanders are in line with your conclusion whether they are truth speakers or liars.

Solution: The claim of A is: $t_A \leftrightarrow (t_B \oplus t_C)$. The claim of B is: $t_B \leftrightarrow t_A$. The claim of C is: $t_C \leftrightarrow t_B$.

Assume, toward a contradiction, that C is a truth speaker: t_C is *true*. Then by the claim of C, B is a truth speaker: t_B is *true*. So by the claim of B, A is a truth speaker: t_A is *true*. So by the claim of A, $t_B \oplus t_C$ is *true*. But since we found that t_B and t_C are both *true*, $t_B \oplus t_C$ is *false*. Contradiction.

So the assumption cannot hold, and hence C is a liar: t_C is *false*. Then by the claim of C, B is a liar: t_B is *false*. So by the claim of B, A is a liar: t_A is *false*.

Check: Since both B and C are liars, $t_B \oplus t_C$ is *false*, so the claim by A is a lie. Since A is a liar, t_A is *false*, so the claim by B is a lie. Since B is a liar, t_B is *false*, so the claim by C is a lie.

3. Conjunctive normal form (CNF) (7 + 7 + 2 points)
Transform the propositional formula $p \lor \neg(q \lor \neg r)$ to conjunctive normal form (CNF) using two different techniques. In both cases describe all the transformation steps.

(a) Rewrite $p \lor \neg(q \lor \neg r)$ to CNF using the algorithm `CNF`.

(b) Transform $p \lor \neg(q \lor \neg r)$ to CNF on the basis of its truth table.

(c) Finally, state whether $p \lor \neg(q \lor \neg r)$ is a tautology, a contradiction or contingent. Explain your answer.

Solution:

(a) $p \lor \neg(q \lor \neg r) \rightsquigarrow p \lor (\neg q \land \neg\neg r) \rightsquigarrow p \lor (\neg q \land r) \rightsquigarrow (p \lor \neg q) \land (p \lor r)$

(b) In the truth table of $p \lor \neg(q \lor \neg r)$, three lines evaluate to *false*:

- p *false*, q *true*, r *true*
- p *false*, q *true*, r *false*
- p *false*, q *false*, r *false*

This yields the CNF $(p \lor \neg q \lor \neg r) \land (p \lor \neg q \lor r) \land (p \lor q \lor r)$.

(c) The formula is contingent. For instance, taking p, q and r all *true* yields the outcome *true*, while taking p, q and r all *false* yields the outcome *false*.

4. Sets (6 + 3 + 6 points)

(a) Consider three sets A, B and C in a universe U of 33 elements. Suppose we know in addition that

$$\#A = 21 \qquad \#B = 13 \qquad \#C = 15$$
$$\#(B \cap C) = 6 \qquad \#(C \cap A) = 11 \qquad \#(A \cap B \cap C) = 4$$
$$\#(A \cup B) = 26.$$

Draw a Venn diagram of the sets A, B and C, and determine the number of elements in every region of your Venn diagram.

(b) Use your Venn diagram from part (a) to determine the following numbers:

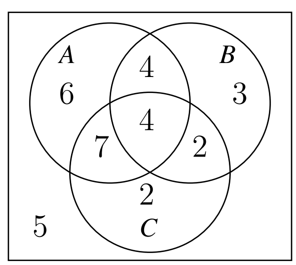
$$\#(B \cup C)', \qquad \#(C \setminus B), \qquad \#(A \cap B).$$

(c) Derive the following equality (for all sets A, B and C), using the algebra of sets:

$$(B \setminus C) \cup (B \cap A) = B \setminus (C \setminus A).$$

Solution:

(a) This is the Venn diagram with the number of elements for each region:



(b) We see from the Venn diagram that

$$\#(B \cup C)' = 11, \qquad \#(C \setminus B) = 9, \qquad \#(A \cap B) = 8.$$

(c)
$$\begin{aligned}
(B \setminus C) \cup (B \cap A) &= (B \cap C') \cup (B \cap A) && \text{(definition)}\\
&= B \cap (C' \cup A) && \text{(distributivity)}\\
&= B \cap (C' \cup (A')') && \text{(involution)}\\
&= B \cap (C \cap A')' && \text{(De Morgan)}\\
&= B \cap (C \setminus A)' && \text{(definition)}\\
&= B \setminus (C \setminus A) && \text{(definition).}
\end{aligned}$$

5. Relations (4 + 6 + 6 points)
Consider the relation R in the set $V := \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ that has the matrix representation

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

where the first row and column correspond to element a, the second row and column to element b, the third row and column to c, and the last row and column to d.

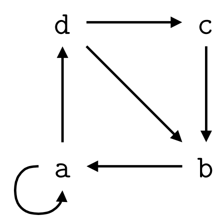
(a) Explicitly list all the elements of R in the curly-bracket notation, and draw the directed graph representation of R.

(b) Is the relation R reflexive? transitive? symmetric? anti-symmetric? Briefly explain your answers.

(c) Determine the relation $R^{-1} \circ R$, explicitly list all its elements in the curly-bracket notation, and draw its directed graph representation.

Solution:

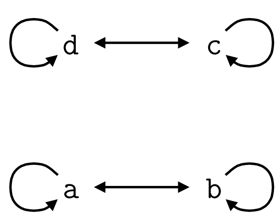
(a) $R = \{\langle \mathtt{a}, \mathtt{a} \rangle, \langle \mathtt{a}, \mathtt{d} \rangle, \langle \mathtt{b}, \mathtt{a} \rangle, \langle \mathtt{c}, \mathtt{b} \rangle, \langle \mathtt{d}, \mathtt{b} \rangle, \langle \mathtt{d}, \mathtt{c} \rangle\}$. The directed graph:



(b) Reflexive? No, because (for example) $\langle \mathtt{b}, \mathtt{b} \rangle \notin R$.
Transitive? No: $\langle \mathtt{b}, \mathtt{a} \rangle \in R$ and $\langle \mathtt{a}, \mathtt{d} \rangle \in R$, but $\langle \mathtt{b}, \mathtt{d} \rangle \notin R$.
Symmetric? No: $\langle \mathtt{b}, \mathtt{a} \rangle \in R$ but $\langle \mathtt{a}, \mathtt{b} \rangle \notin R$.
Anti-symmetric? Yes: for all $x \neq y$ such that $\langle x, y \rangle \in R$ we have that $\langle y, x \rangle \notin R$.

(c) The relation $R^{-1} \circ R$ consists of all pairs $\langle x, y \rangle$ such that for some z, $\langle x, z \rangle \in R$ and $\langle y, z \rangle \in R$. In the picture above this means that you can follow an arrow from x to some point z, and then an arrow *in the opposite direction* from z to y. Hence the relation and its directed graph are:

$$R^{-1} \circ R = \{\langle \mathtt{a}, \mathtt{a} \rangle, \langle \mathtt{a}, \mathtt{b} \rangle, \langle \mathtt{b}, \mathtt{a} \rangle, \langle \mathtt{b}, \mathtt{b} \rangle, \langle \mathtt{c}, \mathtt{c} \rangle, \langle \mathtt{c}, \mathtt{d} \rangle, \langle \mathtt{d}, \mathtt{c} \rangle, \langle \mathtt{d}, \mathtt{d} \rangle\}.$$



6. Ordering relations (8 + 3 + 3 points)

Let A be the subset of $\{0, 1, 2\}^2$ defined by

$$A := \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 0, 2\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle\}.$$

In this task, we consider the Cartesian ordering relation on A induced by the usual partial order $\leq$ on the elements of the set $\{0, 1, 2\}$.

(a) Use the algorithm that you have learned to construct the Hasse diagram that represents the Cartesian ordering relation on A. Explicitly write down the sets G_x and H_x obtained in the construction, and draw the Hasse diagram.

Remark: to simplify the notation, you may write 00 instead of $\langle 0, 0\rangle$, 12 instead of $\langle 1, 2\rangle$, et cetera.

(b) Does the set A have a smallest element according to the Cartesian ordering relation? If so, please give this smallest element. If not, please list all the minimal elements of the set A.

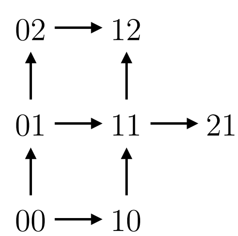
(c) Does the set A have a largest element according to the Cartesian ordering relation? If so, please give this largest element. If not, please list all the maximal elements of the set A.

Solution:

(a) The algorithm gives the following sets G_x and H_x:

$$\begin{aligned}
G_{00} &= \{01, 02, 10, 11, 12, 21\} & H_{00} &= \{01, 10\} \\
G_{01} &= \{02, 11, 12, 21\} & H_{01} &= \{02, 11\} \\
G_{02} &= \{12\} & H_{02} &= \{12\} \\
G_{10} &= \{11, 12, 21\} & H_{10} &= \{11\} \\
G_{11} &= \{12, 21\} & H_{11} &= \{12, 21\} \\
G_{12} &= \{\,\} & H_{12} &= \{\,\} \\
G_{21} &= \{\,\} & H_{21} &= \{\,\}
\end{aligned}$$

The Hasse diagram therefore looks like this:



(b) Yes, as the Hasse diagram shows, $\langle 0, 0\rangle$ is the smallest element of A in the Cartesian order.

(c) No, there is no largest element. As the Hasse diagram shows, both $\langle 1, 2\rangle$ and $\langle 2, 1\rangle$ are maximal elements of the set A in the Cartesian order.