

This exam has 6 pages and 11 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. **Sets and binary relations** ($4 + 4 + 3 + 3 + 3$ points)

- (a) Construct a Venn diagram for each of the following three formulas with the sets A , B , and C in universe U . Clearly denote how the construction is obtained and which area is given by the formula.

$$(A \Delta B) \Delta C, \quad U \setminus (A \cap B' \cap C'), \quad (A' \setminus B)' \cap C.$$

- (b) Prove the equality of the following formula with the laws of algebra for sets.

$$U \setminus ((A \cap B') \cup C') = C \cap (A' \cup B).$$

- (c) Give the inverse relations of the following binary relations in the set *People*.

- i. *IsStudentOf*,
- ii. *LivesInTheSameBuildingAs*,
- iii. *IsInLoveWith*,
- iv. *IsFatherOf*,
- v. *IsVisiting*.

- (d) Simplify the following relations in the composition by one statement.

- i. $IsSiblingOf \circ IsUncleOf$,
- ii. $IsParentOf \circ IsGrandChildOf$ (provide two options).

- (e) Check if the following relations are reflexive and transitive. Motivate your answer.

- i. *HasADifferentVUnetIDThan*,
- ii. *JumpsAtLeastAsHighAs*,
- iii. *IsAsRichAs*,
- iv. *HasAClockThatGivesATimeThatDiffersByOneMinuteAsComparedToTheClockOf*.

2. **Syntax of propositional logic** (3 + 3 points)

- (a) Draw the parse tree of the formula $\neg(p \wedge q \rightarrow r \vee (q \rightarrow r))$.
- (b) Use the parse tree obtained to compute the truth value of the formula in (a) for the valuation that assigns T to p and q , and F to r .

3. **Semantic entailment** (4 points)

Does the semantic entailment below hold or does it not hold?

$$\neg r \rightarrow (p \vee q), r \wedge \neg q \models r \rightarrow q$$

Show clearly how you obtain your answer, and why it settles the question.

4. **Island puzzle** (2 + 4 + 2 points)

On the island of liars and truth speakers everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth). You are on vacation on this island.

You meet two islanders A and B , and want to determine to which groups they belong. You hear that A says: “ B is a truth speaker.” But you also hear that B adds: “It is not true that if A is a liar, then I am a liar.”

- (a) Give two formulas in propositional logic using variables W_A (for ‘ A is a truth speaker’) and W_B (for ‘ B is a truth speaker’) that express the dependency of the statements of A and B on whether they are truth speakers, respectively.
- (b) For each of the islanders A and B determine, by using a truth table for the formulas obtained in (a), whether she/he is a truth speaker or a liar. Check whether the obtained result does indeed make sense, by comparing it with the description of the situation as above.¹
- (c) Which valid semantic implication allows you to draw the conclusions reached?

5. **Conjunctive normal form** (4 points)

By using the rules of the algorithm CNF, rewrite the following formula step by step into a conjunctive normal form:

$$((p \rightarrow q) \rightarrow p) \wedge (p \vee (q \wedge \neg r))$$

¹Note that if the check does not succeed, this is an indication that at least one of the formulas in (a) was not chosen correctly.

6. Relations ($4 + 2 + 2 + 4 + 2$ points)

We are given the following set of numbers

$$\text{Div60} := \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$

(the set of all divisors of 60), with the binary relation *Divides* in *Div60* defined by “ x is a divisor of y ”. It is well known that this relation induces a partial order on the set (one does not need to show this).

- (a) Represent the relation *Divides* in *Div60* by a Hasse diagram. While constructing the Hasse diagram, please list the sets Gx as well.
- (b) Now take subset $A := \{3, 4, 5, 6, 10, 12, 15, 20, 30\}$ of *Div60*. Does A have a smallest element according to the order relation *Divides*? If so, please give this element; if not, please list all minimal elements. Clearly mention if these do not exist either.
- (c) (Follow up on part b) Does A have a largest element according to the order relation *Divides*? If so, please give this element; if not, please list all maximal elements. Clearly mention if these do not exist either.
- (d) Let f be a function that maps the elements of *Div60* to the remainder after dividing by 10, thus $f(x) = x \bmod 10$. Consider the equivalence relation R in the set *Div60*, that is defined by

$$x R y \text{ if and only if } f(x) = f(y).$$

How many different equivalence classes are there?

- (e) Give a full system of representatives for the equivalence relation R .

7. Functions ($3 + 3$ points)

- (a) What does it mean when the typed function $f : A \rightarrow B$ is a total bijection? Give the three characteristics and provide their definition.
- (b) What does it mean when two sets A and B have the same cardinality? Please provide a definition.

8. **Logic circuits and BDDs** ($2 + 3 + 3$ points)

Let $f_1(x, y)$ and $f_2(x, y)$ be two binary Boolean functions that are defined as follows:

$f_1(x, y)$ yields output 0 for all inputs x and y .

$f_2(x, y)$ yields output 0 if and only if x is 0 and y is 1.

- (a) Represent $f_1(x, y)$ and $f_2(x, y)$ by propositional formulas.
- (b) Construct a logic circuit from \neg -, \wedge -, and \vee -gates that has two input channels, one for x and one for y , and two output channels, one for $f_1(x, y)$ and one for $f_2(x, y)$. Indicate directions of information flow.
- (c) Construct a binary decision tree (BDT) that implements the second of these functions, f_2 , and reduce this BDT to a reduced binary decision diagram (BDD).

9. **Binary numbers** ($2 + 1$ points)

- (a) Write 35 as a binary number.
- (b) To the result of (a) add the binary representation of 5 by carrying this out as a binary addition.

10. **Predicate logic** ($(2 + 3 + 3) + 4$ points)

- (a) Translate, based on the translation key:

$C(x)$: x is a clever person $W(x)$: x is a wise person

$K(x, y)$: x knows y ,

the following sentences into formulas of predicate logic:

- (i) Every wise person knows her-/himself.
 - (ii) There are persons who are wise and clever, but not every wise person is clever.
 - (iii) Every wise person knows somebody who knows a clever person.
- (b) Describe a model \mathcal{M} with a two-person domain $A = \{p_1, p_2\}$ such that all of the sentences in (a) (and the corresponding formulas) are true in \mathcal{M} .

For such a model \mathcal{M} appropriate interpretations $C^{\mathcal{M}}$, $W^{\mathcal{M}}$, and $K^{\mathcal{M}}$ in \mathcal{M} of the predicate symbols C , W , and K have to be defined. A drawing may suffice, but these 3 interpretations must be clearly specified.

11. Induction and Recursion (*4 + 4 points*)

(a) Consider a sequence of real-valued numbers $(t_n)_{n=0}^{\infty}$ defined recursively by

$$t_0 := 0, \quad t_{n+1} := t_n + 3n + 2.$$

- i. Calculate the terms t_1, \dots, t_6 of this sequence.
- ii. Prove by mathematical induction that

$$t_n = \frac{1}{2}n(3n + 1), \quad n \geq 0.$$

(b) In the set $V := \{a, b, c, d\}$ we have a binary relation R defined by

$$R := \{(a, b), (a, c), (b, c), (c, b), (c, d), (d, a)\}.$$

- i. Depict the relation R as a directed graph.
- ii. List all elements of the relation $R \circ R$.

Algebra for sets

Commutativity:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Idempotence:

$$A \cup A = A$$

$$A \cap A = A$$

Associativity:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Complement:

$$A \cup A' = U$$

$$A \cap A' = \emptyset$$

Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

DeMorgan's Laws:

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Identities:

$$A \cup U = U \text{ en } A \cup \emptyset = A$$

$$A \cap U = A \text{ en } A \cap \emptyset = \emptyset$$

Involution:

$$(A')' = A$$