Department of Computer Science
VU University Amsterdam

Exam *Logic and Sets*
March 27, 2015, 12:00–14:45

This exam has 6 pages and 11 exercises.
The result will computed as (total number of points plus 10) divided by 10.
Answers may be given in either English or Dutch.
Please motivate all answers!

1. **Sets and binary relations** *(4 + 4 + 3 + 3 + 3 points)*

   (a) Construct a Venn diagram for each of the following three formulas with the sets A, B, and C in universe U. Clearly denote how the construction is obtained and which area is given by the formula.

   $$(A \,\Delta\, B) \cap C, \qquad U \setminus (A' \cap B' \cap C), \qquad (A \setminus C)' \cap B.$$

   (b) Prove the equality of the following formulas with the laws of algebra for sets (which are supplied on page 6).

   $$U \setminus \big((A \cup B)' \cap C'\big) = A \cup B \cup C.$$

   (c) Give the inverse relations of the following binary relations.

   i. *IsTeacherOf*,
   ii. *HasAnEqualToOrHigherGradeThan*,
   iii. *LivesOnTheSameFloorAs*,
   iv. *IsCreatorOf*,
   v. *IsDestinationOfFlightWithOrigin*.

   (d) Simplify the following relations in the composition by one statement.

   i. *IsParentOf* $\circ$ *IsBrotherOf*,
   ii. *IsGrandParentOf* $\circ$ *IsDaughterOf* (provide two options).

   (e) Check if the following relations are reflexive and transitive. Motivate your answer.

   i. *IsCloserThanFiveKilometersOf*,
   ii. *JumpsHigherThan*,
   iii. *HasTheSameGradeAs*,
   iv. *LivesTwoKilometersApartFrom*.

2. **Syntax of propositional logic** *(3 + 3 points)*

   (a) Draw the parse tree of the formula $\neg(s \to \neg(p \to (q \vee \neg s)))$.

   (b) Use the parse tree obtained to compute the truth value of the formula in (a) for the valuation that assigns `F` to p, and `T` to q and s.

3. **Semantic entailment** *(4 points)*
   Investigate whether the following semantic statement holds or does not hold:

$$p \to (\neg q \vee r),\ \neg r \ \vDash\ \neg q \to \neg p$$

   Show clearly how you obtain your answer.

4. **Island puzzle** *(2 + 5 points)*
   On the island of liars and truth speakers everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth). You are on vacation on this island.

   You meet two islanders A and B, and want to determine to which groups they belong. You hear that A says: "I am not different from B what concerns telling the truth or not." [1] But you also hear that B protests: "Beware, not both of us speak the truth."

   (a) Give two formulas in propositional logic using variables W_A (for 'A is a truth speaker') and W_B (for 'B is a truth speaker') that express the dependency of the statements of A and B on whether they are truth speakers, respectively.

   (b) For each of the islanders A and B determine whether she/he is a truth speaker or a liar. Motivate your answer (either by arguing verbally, or by using a truth table for the formulas obtained in (a)).

5. **Conjunctive normal form** *(4 points)*
   Transform the following formula into a conjunctive normal form:

$$\neg(q \wedge (\neg p \to q)) \wedge (q \to \neg p)$$

   Show clearly how you obtain your result.

[1] Rephrased, A says: "Either both of us are truth speakers, or both of us are liars.", or: "I am a liar if and only if B is a liar".

6. **Relations** *(4 + 2 + 2 + 4 + 2 points)*

We are given the following set of pairs as subset of $\{a, b, c, d\} \times \{a, b, c, d\}$

$$V := \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\},$$

with the Cartesian ordering CART with respect to the lexicographic ordering on the components. It is well known that this relation induces a partial order on the set (you do not need to show this).

(a) Represent the relation CART by a Hasse diagram. While constructing the Hasse diagram, please list the sets Gx as well.

(b) Now take subset $A := \{(a, b), (b, b), (c, c), (c, d), (d, c)\}$ of V. Does A have a smallest element according to the order relation CART? If so, please give this element; if not, please list all minimal elements. Clearly mention if these do not exist either.

(c) (Follow up on part b) Does A have a largest element according to the order relation CART? If so, please give this element; if not, please list all maximal elements. Clearly mention if these do not exist either.

(d) Let f be a function that maps the elements of $\{a, b, c, d\}$ to specific numbers, thus $f(a) = 1$, $f(b) = 2$, $f(c) = 3$, and $f(d) = 4$. Consider the equivalence relation R in the set V, that is defined by

$$(x_1, x_2)\, R\, (y_1, y_2) \text{ if and only if } |f(x_1) + f(x_2)| = |f(y_1) + f(y_2)|.$$

How many different equivalence classes are there?

(e) Give a full system of representatives for the equivalence relation R.

7. **Functions** *(2 + 4 points)*

(a) What does it mean when the typed function $f : A \to B$ is total, injective, and surjective? Please define the three concepts.

(b) Consider the typed function $f : A \to B$ with $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ given by $f(x) := x^2$. Give non-empty sets A and B such that the function f is a total bijection.

8. **Logic circuits and BDDs** *(5 + 3 points)*
Let $f_1(x,y)$ and $f_2(x,y)$ be two binary Boolean functions that are defined as follows:

$f_1(x,y)$ yields output `1` for all inputs x and y.

$f_2(x,y)$ yields output `1` if and only if both x and y are `0`.

(a) Construct a logic circuit from $\neg$-, $\wedge$-, and $\vee$-gates that has two input channels, one for x and one for y, and two output channels, one for $f_1(x,y)$ and one for $f_2(x,y)$.

(b) Construct a reduced binary decision diagram (BDD) that implements the second of these functions, f_2.

9. **Binary numbers** *(2 + 2 points)*

(a) What is the result of the following binary addition?

$$\begin{array}{r} 1\;1\;0\;0\;1\;1\;1 \\ +\quad\;\; 1\;1\;1\;1\;0\;1 \\ \hline \end{array}$$

(b) Write 25 as a binary number.

10. **Predicate logic** *((2 + 3 + 3) + 4 points)*

(a) Translate, based on the translation key:

$C(x)$: x is a clever person $\qquad$ $W(x)$: x is a wise person

$LL(x)$: x has learned logic $\qquad$ $K(x,y)$: x knows y ,

the following sentences into formulas of predicate logic:

(i) Everybody who has learned logic is clever.

(ii) Every wise person knows herself/himself, but there are clever persons who do not know themselves.

(iii) Every clever person knows some wise person.

(b) Describe a model $\mathcal{M}$ with a two-person domain $A = \{p_1, p_2\}$ such that:

▷ all of the sentences in (a) (and the corresponding formulas) are true in $\mathcal{M}$;

▷ there is at least one clever person in $\mathcal{M}$.

(For such a model $\mathcal{M}$ appropriate interpretations $C^{\mathcal{M}}$, $W^{\mathcal{M}}$, $LL^{\mathcal{M}}$, and $K^{\mathcal{M}}$ in $\mathcal{M}$ of the predicate symbols C, W, LL and K have to be defined. A drawing may suffice, but these 4 interpretations must be clearly specified.)

11. **Induction and Recursion** *(4 + 4 points)*

(a) Consider a sequence of real-valued numbers $(t_n)_{n=0}^{\infty}$ defined recursively by

$$t_0 := 0, \qquad t_{n+1} := t_n + 2n + 3.$$

i. Calculate the terms $t_1, \ldots, t_6$ of this sequence.

ii. Prove by mathematical induction that

$$t_n = n(n+2), \quad n \geq 0.$$

(b) In the set $V := \{1, 2, 3, 4\}$ we have a binary relation R defined by

$$R := \{(1,2), (1,3), (2,3), (3,2), (3,4), (4,1)\}.$$

i. Depict the relation R as a directed graph.

ii. List all elements of the relation $R \circ R$.

Algebra for sets

Commutativity:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Idempotence:

$$A \cup A = A$$
$$A \cap A = A$$

Associativity:

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Complement:

$$A \cup A' = U$$
$$A \cap A' = \emptyset$$

Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

DeMorgan's Laws:

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

Identities:

$$A \cup U = U \text{ en } A \cup \emptyset = A$$
$$A \cap U = A \text{ en } A \cap \emptyset = \emptyset$$

Involution:

$$(A')' = A$$