
This exam has 5 pages and 8 exercises. The result will computed as (total number of points plus 10) divided by 10.
Answers may be given in either English or Dutch.

1. **Sets and binary relations** *(4 + 4 + 3 + 3 + 3 points)*

   (a) Construct a Venn diagram for each of the following three formulas. Clearly denote how the construction is obtained and which area is given by the formula.

   $$(A \cap B) \cup (A \cap C) \cup (B \cap C), \qquad (A' \setminus C) \cap B', \qquad (B \cup C)' \cap A'.$$

   (b) Prove the equality of the following formula with the laws of algebra for sets.

   $$(A' \setminus C)' \cap B = (A \cap B) \cup (B \cap C).$$

   (c) Give the inverse relations of the following binary relations.

   i. *HasTheSameGradeAs*,
   ii. *IsSubsetOf*,
   iii. *IsGrandparentOf*,
   iv. *IsMotherOf*,
   v. *IsElementOfTheEquivalenceClassOf*.

   (d) Simplify the following relations in the composition.

   i. $\mathit{IsChildOf} \circ \mathit{IsSiblingOf}$,
   ii. $\mathit{IsChildOf} \circ \mathit{IsMarriedTo}$.

   (e) Check if the following relations are reflexive and transitive. Motivate your answer.

   i. *IsGreaterThan*,
   ii. *IsNotADivisorOf*,
   iii. *DiffersByAtLeastTwoDaysFrom*,
   iv. *HasADivisorInCommonWith*.

2. **Relations** *(4 + 2 + 2 + 4 + 2 points)*

We are given the following set of numbers

$$Div24 := \{1, 2, 3, 4, 6, 8, 12, 24\}$$

(the set of all divisors of 24), with the binary relation *Divides* in $Div24$ defined by "x is a divisor of y". It is well known that this relation induces a partial order on the set (one does not need to show this).

(a) Represent the relation *Divides* in $Div24$ by a Hasse diagram. While constructing the Hasse diagram, please list the sets Gx as well.

(b) Now take subset $A := \{2, 4, 6, 12\}$ of $Div24$. Does A have a smallest element according to the order relation *Divides*? If so, please give this element; if not, please list all minimal elements. Clearly mention if these do not exist either.

(c) (Follow up on part b) Does A have a largest element according to the order relation *Divides*? If so, please give this element; if not, please list all maximal elements. Clearly mention if these do not exist either.

(d) Consider the equivalence relation R in the set $Div24$, that is defined by

$w_1 R w_2$ if and only if the text representation of w_1 and w_2 end with the same letter,

e.g., $1\,R\,3$ since the text representation of 1 = one and 3 = three both end with 'e'. How many different equivalence classes are there?

(e) Give a full system of representatives for the equivalence relation R.

3. **Syntax** *(3 + 3 points)*

(a) Draw the parse tree of the formula $\neg p \wedge q \rightarrow \neg (r \vee p)$

(b) Compute in the parse tree bottom-up the truth value of this formula, given the truth values T, F, T, for p, q, r, respectively.

4. **Logic** *(4 + 2 + 3 + 3 points)*

(a) Investigate the validity of the semantic entailment

$$(p \rightarrow q) \rightarrow p \models p$$

(Show clearly how you get to your answer.)

(b) What does it mean to say: "the set of formulas $\{\phi_1, \phi_2\}$" is satisfiable?

(c) Is the set $\{(p \rightarrow q) \rightarrow p,\ \neg p\}$ satisfiable?
(Motivate your answer.)

(d) Give a CNF for the formula $(p \rightarrow q) \rightarrow p$

5. **Functional completeness** *(3 + 3 points)*

   (a) What does it mean to say that a system of connectives is *functional complete* (or *adequate*)?

   (b) Is the system of connectives $\{\vee, \rightarrow\}$ adequate?
   Give a short motivation of your answer.

6. **Binary arithmetic** *(2 + 3 points)*

   (a) Write *15* as binary number.

   (b) What is the result of the following binary addition?

$$\begin{array}{ccccccc} 1 & 0 & 1 & 1 & 1 & & \\ & 1 & 0 & 1 & 1 & + & \\ \hline \end{array}$$

7. **Boolean functions** *(4 + 3 points)*

   Given is the boolean function $f(x, y)$, having value 1 if at least one of the two variables x, y is 0, and value 0 otherwise.

   (a) Draw a logic circuit with $\wedge$-, $\vee$- and $\neg$-gates for $f(x, y)$.

   (b) Draw a reduced BDD for the function $f(x, y)$

8. **Predicate logic** *(3 + 3 + 3 points)*

   Translate the following sentences to predicate logic using the specification:

Wx:	x works
Bxy:	x is brother of y
Hx:	x is at home
a:	Anna

   (a) Some of Anna's brothers are at home

   (b) Anna works, but her brothers do not work

   (c) If all her brothers are at home, then Anna does not work

9. **Functions** *(2 + 4 points)*

   (a) What is the difference between the image and the range of a typed function $f : A \to B$.

   (b) Let $A := \{a, b, c, d, e\}$ and $V := A \times A$. Given is a function $value : V \to \{1, \ldots, 10\}$ of which the value is determined by the sum of the letter values ($a = 1, b = 2, c = 3, d = 4, e = 5$), e.g., $value(ab) = 1 + 2 = 3$. Is this function total? Is this function surjective? Please provide arguments.

10. **Induction and Recursion** *(4 + 4 points)*

   (a) Consider a sequence of real-valued numbers $(t_n)_{n=1}^{\infty}$ defined recursively by

$$t_1 := 1, \qquad t_{n+1} := t_n + (3n - 2).$$

      i. Calculate the terms $t_2, \ldots, t_6$ of this sequence.

      ii. Prove by mathematical induction that

$$t_n = \frac{n(3n-1)}{2}, \quad n \geq 1.$$

   (b) In the set $V := \{1, 2, \ldots, 6\}$ we have a binary relation R defined by

$$R := \{< 1, 2 >, < 2, 3 >, < 3, 1 >, < 4, 5 >, < 5, 6 >, < 6, 4 >\}.$$

      i. Depict the relation R as a direct graph as well as $R \circ R$.

      ii. Describe or depict the transitive closure of R.

Algebra for sets

Commutativity:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Idempotence:

$$A \cup A = A$$
$$A \cap A = A$$

Associativity:

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Complement:

$$A \cup A' = U$$
$$A \cap A' = \emptyset$$

Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

DeMorgan's Laws:

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

Identities:

$$A \cup U = U \text{ en } A \cup \emptyset = A$$
$$A \cap U = A \text{ en } A \cap \emptyset = \emptyset$$

Involution:

$$(A')' = A$$
