Logic and Modelling
Vrije Universiteit Amsterdam
Period 5, 2021–2022
Final exam (v2)
2 June 2022
8:30–11:15

Name: ____________________

This exam contains 8 pages (including this cover page) and 7 questions. There are 90 possible points in total.

During this exam, you are allowed to reference

- the *Logic and Proof* and *Logic in Computer Science* textbooks;
- the official lecture slides and supplementary material;
- material from the exercise classes;
- your own notes from the course;
- other written material.

You are **not** allowed to

- have any kind of communication during the exam;
- use your smartphone or some other device.

Write your answers to separate questions on separate sheets of paper. Make your solutions stand out clearly. Please do not submit your scratch work.

In case you notice an ambiguity or errors in an exam question, we strongly recommend that you work things out yourselves, stating explicitly any ambiguity or error and explaining how you interpret or repair the question. The more explicit you are, the easier it will be for us to grade the question afterwards.

Grade table (for teacher use only)

Question:	1	2	3	4	5	6	7	Total
Points:	14	20	12	15	11	10	8	90
Score:								

1. This task concerns validity and (un)satisfiability in propositional and predicate logic.

   For this task, the answers suffice. No justification is necessary.

   Determine for each of the following formulas whether it is valid, satisfiable (but not valid), or unsatisfiable.

   (a) (2 points) $P \to \neg P$

   (b) (2 points) $(\forall x\, P(x)) \vee (\forall x\, \neg P(x))$

   (c) (2 points) $((P \to P) \to Q) \to Q$

   (d) (2 points) $(\forall x\, \forall y\, (P(x,y) \to P(y,x))) \wedge (\exists x\, \neg P(x,x))$

   Solution:

   (a) satisfiable
   (b) satisfiable
   (c) valid
   (d) satisfiable

   Determine for each of the following sets of formulas whether the set is consistent or inconsistent:

   (e) (3 points) $\{\, \forall x\, P(x,x),\ \exists x\, \exists y\, \neg P(x,y) \,\}$

   (f) (3 points) $\{\, \forall x\, (\neg Q(x) \vee R(x)),\ \exists x\, Q(x),\ \forall x\, \neg R(x) \,\}$

   Solution:

   (e) consistent
   (f) inconsistent

2. This task concerns natural deduction and countermodels in propositional and (first-order) predicate logic.

   Determine for each of the given semantic entailments whether it is valid.

   - For the valid entailments, give a derivation using natural deduction. Please label the rules you use with their short names (e.g., $\forall$E).
   - For the invalid entailments, give a countermodel. In particular, if the entailment is in propositional logic, give a truth assignment of the variables that shows the entailment is invalid.

   (a) (5 points) $P \to Q,\, P \to \neg Q \models \neg P$

   (b) (5 points) $P \to \neg Q,\, Q \to \neg P \models \neg Q$

   (c) (5 points) $\forall x\, P(x), \forall x\, Q(x) \models \forall x\, \forall y\, (P(x) \lor Q(y))$

   (d) (5 points) $\forall x\, (P(x) \to \neg Q(x)), \forall x\, (\neg Q(x) \to R(x)) \models \forall x\, (P(x) \to R(x))$

Solution:

(a) valid:

$$\dfrac{\dfrac{\dfrac{P \to \neg Q \qquad \dfrac{}{P}\,1}{\neg Q} \qquad \dfrac{P \to \neg Q \qquad \dfrac{}{P}\,1}{Q}}{\bot}\,\neg\mathrm{E}}{\neg P}\,\neg\mathrm{I},\,1$$

(b) invalid:

Take P to be false and Q to be true. (Then the two assumptions are true, but the conclusion is false.)

(c) valid

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x\, P(x)}{P(x)}\,\forall\mathrm{E}}{P(x) \lor Q(y)}\,\lor\mathrm{I}_\ell}{\forall y\, (P(x) \lor Q(y))}\,\forall\mathrm{I}}{\forall x\, \forall y\, (P(x) \lor Q(y))}\,\forall\mathrm{I}$$

(d) valid

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x\, (\neg Q(x) \to R(x))}{\neg Q(x) \to R(x)}\,\forall\mathrm{E} \qquad \dfrac{\dfrac{\forall x\, (P(x) \to \neg Q(x))}{P(x) \to \neg Q(x)}\,\forall\mathrm{E} \qquad \dfrac{}{P(x)}\,1}{\neg Q(x)}\,\to\mathrm{E}}{R(x)}\,\to\mathrm{E}}{P(x) \to R(x)}\,\to\mathrm{I},\,1}{\forall x\, (P(x) \to R(x))}\,\forall\mathrm{I}$$

3. This task concerns the translation of natural language sentences to predicate logic.

   The domain is the set that contains every person on earth. The meaning of the symbols is:

a : Anne	$G(x)$: x is a good teaching assistant (TA)
v : Visa	$T(x,y)$: x is taller than y
	$C(x,y)$: x and y have the same hair color

   Translate the following sentences into formulas of predicate logic:

   (a) (3 points) Nobody is taller than everybody else.

   (b) (3 points) Everybody who has the same hair color as Anne also has the same hair color as Visa.

   (c) (3 points) Exactly one person is a good TA.

   (d) (3 points) Both Anne and Visa are good TAs, and nobody else is a good TA.

   Solution:

   (a) $\neg\exists x\,\forall y\,(x \neq y \rightarrow T(x,y))$

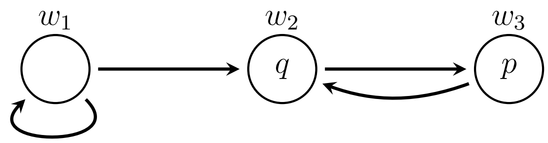
   (b) $\forall x\,(C(x,a) \rightarrow C(x,v))$

   (c) $\exists x\,G(x) \wedge (\forall y\,(G(y) \rightarrow y = x))$

   (d) $G(a) \wedge G(v) \wedge (\forall x\,(G(x) \rightarrow x = a \vee x = v))$

4. This task concerns basic modal logic and Kripke structures.

   Given is the Kripke model $\mathcal{M} = (W, R, L)$ as drawn in the following figure:



   Determine *for each* world x whether or not the following statements hold. You do not have to explain your answer.

   (a) (3 points) $\mathcal{M}, x \models \Diamond q \to p$

   (b) (3 points) $\mathcal{M}, x \models \Box q$

   Solution:

   (a) w_1: no, w_2: yes, w_3: yes

   (b) w_1: no, w_2: no, w_3: yes

   We now consider the underlying frame $\mathcal{F} = (W, R)$. Determine whether the following statements hold. If yes, give an argument why. If no, give a counterexample.

   (c) (3 points) $\mathcal{F} \models \Diamond\Diamond p \to p$

   (d) (3 points) $\mathcal{F} \models \Box p \to \Diamond p$

   Solution:

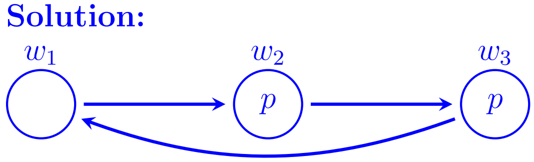
   (a) invalid:
   Consider the labeling as given above. At w_1, we have $\Diamond\Diamond p$ but not p.

   (b) valid:
   Every world has at least *one* out-arrow. Hence, if p holds for all next nodes, it holds for at least *one*.

   Draw a Kripke model with three worlds where the following sentence is true at two of the worlds and false at the third. Clearly identify at which worlds the sentence is true.

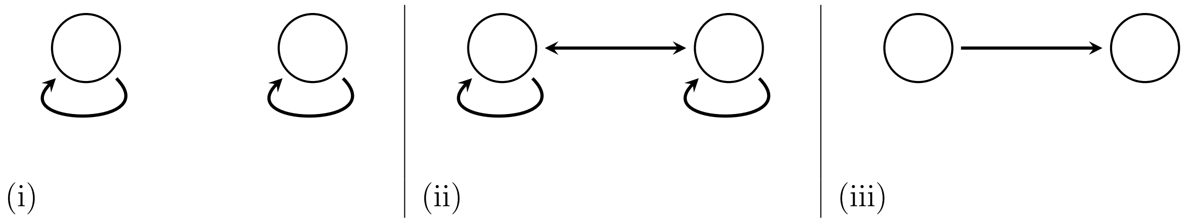
   (e) (3 points) $\Diamond p \lor \Box q$

   Solution:



   The sentence is true in worlds w_1 and w_2, because $\Diamond p$ is then true. ($\Box q$ is false in all three worlds.)

5. This task is about correspondence of formulas and frame properties in modal logic.



(a) (3 points) Let $\mathcal{M} = (W, R, L)$ be a Kripke model. Suppose that $\mathcal{M} \models \Diamond p \wedge \Diamond \neg p$. Which of the frames (i)–(iii) above could be the underlying frame $\mathcal{F} = (W, R)$? Briefly explain your answer.

Solution:

Frame (ii) could be it, with p at w_1 and $\emptyset$ at w_2 or vice versa.

Frame (i) and (iii) cannot be it, because their worlds do not have two successors, which is necessary to have simultaneously $\Diamond p$ and $\Diamond \neg p$ at a world.

(b) (3 points) For each Kripke frame (i)–(iii) above, is the frame symmetric? Is the frame transitive?

Solution:

(i) symmetric, reflexive

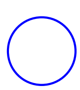
(ii) symmetric, reflexive

(iii) not symmetric, not reflexive

(c) (5 points) The formula $q \to \Diamond q$ does *not* characterize the frame property of symmetry. Show this by presenting a symmetric frame in which $q \to \Diamond q$ is invalid. Give a short explanation.

Solution:

Consider this frame:



Consider the labeling that assigns q to the single world. Then q is true at that world, but $\Diamond q$ is false.

6. This task concerns metatheorems of predicate logic and expressibility of properties in predicate logic.

   (a) (4 points) We saw in class that the formula

$$\varphi_n = \exists x_1 \ldots \exists x_n \left(\bigwedge_{1 \le i < j \le n} x_i \neq x_j\right)$$

   can be used to characterize models that have at least n element. Explain (1) why φ_n is true in any model with at least n elements and (2) why any model of φ_n has at least n elements, in both cases by referring to the semantics of predicate logic.

   Solution:

   Consider a model $\mathcal{M}$ with at least n distinct elements $a_1, \ldots, a_n$. When interpreting φ_i, we can take $a_1, \ldots, a_n$ as the witnesses for $x_1, \ldots, x_n$, respectively. Then we have $a_i \neq a_j$ for every $1 \le i < j \le n$.

   Conversely, consider a model $\mathcal{M}$ of φ_n. For $\mathcal{M}$ to be a model, there must exist witnesses $a_1, \ldots, a_n$ for $x_1, \ldots, x_n$, respectively. Moreover, for each $x_i \neq x_j$ to be interpreted as true, each pair $a_i \neq a_j$ must be true. This means the witnesses $a_1, \ldots, a_n$ are all distinct, and hence since $\{a_1, \ldots, a_n\} \subseteq A$, we have $|A| \ge n$.

   (b) (6 points) Alyssa P. Hacker claims that she has found a countermodel of the first-order formula $\varphi = (\forall x\, \forall y\, R(x, y)) \to (\forall x\, R(x, x))$—that is, a model $\mathcal{M}$ such that $\mathcal{M} \not\models \varphi$. However, her countermodel is very complicated, and you do not want to read it.

   Convince her, using a metatheorem (or several metatheorems) of predicate logic, that her countermodel must be wrong.

   Solution:

   We will exhibit a derivation of ϕ without assumptions. By the soundness theorem of predicate logic, $\models \phi$. Hence $\mathcal{M} \models \phi$.

   Here is the derivation:

$$\begin{array}{ll}
\overline{\forall x\, \forall y\, R(x,y)} & 1 \\
\overline{\forall y\, R(x,y)} & \forall\mathrm{E} \\
\overline{R(x,x)} & \forall\mathrm{E} \\
\overline{\forall x\, R(x,x)} & \forall\mathrm{I} \\
\overline{(\forall x\, \forall y\, R(x,y)) \to (\forall x\, R(x,x))} & \to\mathrm{I},\ 1
\end{array}$$

7. This task concerns decision problems and their solvability (decidability).

   (a) (2 points) For each problem, write whether it is decidable or undecidable. You do not need to explain your answer.

      1. the Post correspondence problem
      2. the validity problem in predicate logic

   Solution:

   1. undecidable
   2. undecidable

   (b) (3 points) Consider the problem of taking a natural number n and returning the smallest number $p > n$ such that p is prime.

   Is this a decision problem? If it is a decision problem, is it decidable? If it is not a decision problem, why not?

   Solution:

   This is not a decision problem because the answer is not of the form yes or no.

   (c) (3 points) Give an instance of the Post correspondence problem that has a solution. Give the solution.

   Solution:

   Take $\{a = a\}$ as input. Then there is the trivial solution $a = a$.