Logic and Modeling **Name:** ______________________
Period 5, 2021
Exam
May 27, 2021
Time Limit: 180 Minutes including submission

This exam contains 9 pages (including this cover page) and 8 questions.
There are 100 possible points in total.

During this exam, you are allowed to reference:

- The *Logic and Proof* and *Logic in Computer Science* textbooks.
- The official lecture slides and supplementary material.
- Material from the exercise classes.
- Your own notes from the course.
- Other "static" material online: this means material that does not change and was written before the exam.

You are **not** allowed to:

- Have any kind of communication during the exam.
- Use any "calculator" tool that claims to solve logic problems automatically. (These things are not reliable: there are many, many equivalent ways to present logics, and each such tool is specific to one presentation. We will notice if you use a tool like this.)
- Try to look up answers using Google or other search engines. The exam is designed to make this hard, but please don't test it.

This has been a very unusual and complicated period. To avoid making it even more complicated, we are putting a lot of trust in you with this take home exam. Please do not abuse our trust!

Write your answers on separate sheets of paper. You will submit each answer separately on Canvas. Do not submit your scratch work. Make your solutions stand out clearly.

Happy proving!

Grade Table (for teacher use only)

Question:	1	2	3	4	5	6	7	8	Total
Points:	6	12	20	12	12	10	17	11	100
Score:									

1. (6 points) **IMPORTANT**: your exam will not be graded if you do not submit this.

   The VU has released the following statement of integrity for online exams.

   > I hereby declare, that I understand that taking an online exam during this corona crisis is an emergency measure to prevent study delays as much as possible. I know that fraud control will be tightened and I also realize that a special appeal is now being made to trust my integrity. With this statement, I promise to:
   >
   > - make this exam completely on my own,
   > - do not consult sources, other than explicitly mentioned in the examination,
   > - make myself available for any oral explanation of my answers.

   Please write the following statement and sign your name below it.

   > I have read the statement of integrity and exam rules, and followed them while taking this exam.

2. This task concerns validity and (un)satisfiability in propositional and predicate logic.

   For this task, the answers suffice. No justification is necessary.

   Determine for each of the following formulas whether it is valid, satisfiable (but not valid), or unsatisfiable.

   (a) (2 points) $((q \to \neg p) \to q) \to q$

   (b) (2 points) $\forall x(\forall y P(x, y) \to \forall y P(y, x))$

   (c) (2 points) $\neg s \land (t \to s) \land (\neg s \to t)$

   Determine for each of the following sets of formulas whether the set is consistent or inconsistent:

   (d) (3 points) $\{\ \forall y(P(y) \lor Q(y)),\ \neg P(a),\ \neg Q(b)\ \}$

   (e) (3 points) $\{\ \forall x \forall y(R(y, x) \to R(x, y)),\ \exists x R(x, x),\ \exists x \neg R(x, x)\ \}$

3. This task concerns natural deduction and countermodels in propositional and first order logic.

   Determine for each of the given semantic implications whether it is valid.

   - For the valid implications, give a derivation using natural deduction.
   - For the invalid implications, give a counter model. (In the case of propositional logic, give a truth assignment of the variables that shows the implication is invalid.)

   (a) (5 points) $\models (\neg p \vee \neg q) \to \neg(p \wedge q)$

   (b) (5 points) $\forall x \forall y (D(y) \to E(x)) \models \exists z D(z) \to \forall x E(x)$

   (c) (5 points) $p \to q,\ t \to q \models (p \vee t) \vee q$

   (d) (5 points) $\forall x \forall y (R(x, y) \to x = y) \models R(c, c)$

4. This task concerns the translation of natural language sentences to predicate logic.

   The domain is the set of planets in the galaxy. The meaning of the symbols is:

   e : Earth
   $L(x)$: x supports human life
   $S(x)$: x has a solid surface
   $B(x, y)$: x is bigger than y
   $O(x, y)$: x and y orbit the same sun

   Translate the following sentences into formulas of predicate logic:

   (a) (3 points) Some planet is bigger than every other planet.

   (b) (3 points) Earth has a solid surface, but another planet with a solid surface is bigger than Earth.

   (c) (3 points) No planet that orbits the same sun as Earth supports human life.

   (d) (3 points) At most two planets support human life.

5. This task concerns meta-theorems of predicate logic, and expressibility of properties in predicate logic.

   (a) (6 points) Here is an argument for why model finiteness is not expressible by a set of formulas in first order logic.

   > Suppose there is a set of formulas Γ such that for every model $\mathcal{M}$,
   >
   > $$\mathcal{M} \models \Gamma \Leftrightarrow \mathcal{M} \text{ has finite domain.}$$
   >
   > Consider the set $\Delta = \Gamma \cup \{\phi_2, \phi_3, \phi_4, \ldots\}$, where ϕ_i expresses the sentence "$\mathcal{M}$ has at least i elements in its domain."
   >
   > (1) The set Δ is consistent: ...
   >
   > (2) The set Δ is inconsistent: ...
   >
   > This is a contradiction, and therefore there is no such set of sentences Γ.

   Fill in the two missing arguments. Explain precisely why:

   1. Δ is consistent, and
   2. Δ is inconsistent.

   Name any theorems you use, and explain why those theorems apply in this situation.

   (b) (6 points) I claim that I have a natural deduction proof of the first order formula $\forall x \forall y R(x, y)$. My proof is very long and you don't want to read it.

   Convince me, using a metatheorem (or metatheorems) of predicate logic, that there is a mistake in my proof.

6. This task concerns decision problems and their solvability (decidability).

   (a) (2 points) For each problem, write whether it is decidable or undecidable. You do not need to explain your answer.

      1. The provability problem for first order logic
      2. The validity problem for propositional logic

   (b) (4 points) Consider the problem of taking a formula of propositional logic and listing all of the truth assignments that make the formula true.

      Is this a decision problem? If it is a decision problem, is it decidable? If it is not a decision problem, why not?

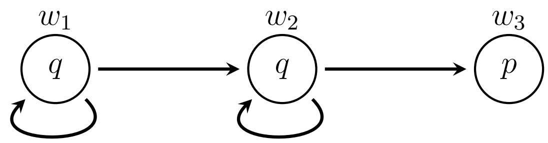
   (c) (4 points) The general *termination problem*, checking whether an arbitrary program terminates with a certain input, is undecidable.

      Suppose I have a limited programming language that is only able to implement programs that terminate. (No *while* loops in this language!) Any program $P(n)$ in this language, taking a natural number as input, must terminate on every input.

      The termination problem for this new language is clearly decidable: for every program, the answer is "yes, it terminates." Explain, in your own words, why the proof of the undecidability of the general termination problem does not apply in this case. What step of the proof fails to hold?

7. This task concerns basic modal logic and Kripke structures.

   Given is the Kripke model $\mathcal{M} = (W, R, L)$ as drawn in the following figure:

   Determine *for each* world x whether or not the following statements hold. You do not have to explain your answer.

   (a) (3 points) $\mathcal{M}, x \models \Diamond p \rightarrow p$

   (b) (3 points) $\mathcal{M}, x \models \Box\Box q$

   We now consider the underlying frame $\mathcal{F} = (W, R)$. Determine whether the following statements hold. If yes, give an argument why. If no, give a counterexample.

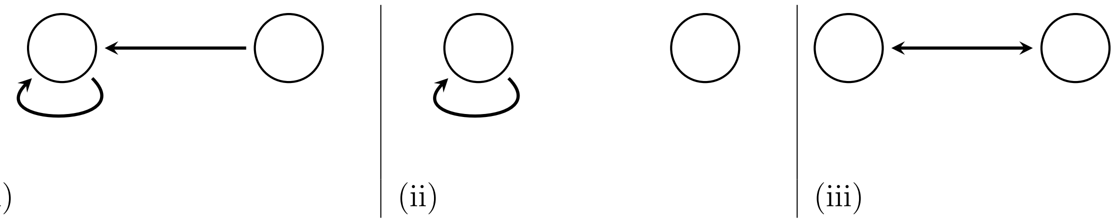
   (c) (4 points) $\mathcal{M} \models \Diamond\Diamond q \rightarrow \Diamond q$

   (d) (4 points) $\mathcal{M} \models \Box p \rightarrow \Box q$

   Draw a Kripke model with three worlds, where the following sentence is true at two of the worlds and false at the third. Clearly identify at which worlds the sentence is true.

   (e) (3 points) $\Diamond p \wedge \Diamond q$

8. This task is about correspondence of formulas and frame properties in modal logic.



(i) (ii) (iii)

(a) (3 points) I have a Kripke model $\mathcal{M} = (W, R, L)$. I tell you that $\mathcal{M} \models \Box p \wedge \Box \neg p$. How many of the frames (i)–(iii) above could be the underlying frame $\mathcal{F} = (W, R)$? Briefly explain your answer.

(b) (3 points) For each Kripke frame (i)–(iii) above, is the frame symmetric? Is the frame transitive?

(c) (5 points) The formula $\Diamond p \to \Box p$ does *not* characterize the frame property of functionality (the property that every world has exactly one successor).

Show this by providing a counterexample, and give a short explanation of why your counterexample works.