Logic and Modelling
Period 5, 2020
Practice Exam
May 8, 2020
Time Limit: 150 Minutes

Name: ______________________

This exam contains 10 pages (including this cover page) and 9 questions. There are 100 possible points in total.

During this exam, you are allowed to reference:

- The *Logic and Proof* and *Logic in Computer Science* textbooks.
- The official lecture slides and supplementary material.
- Material from the exercise classes.
- Your own notes from the course.
- Other "static" material online: this means material that does not change and was written before the exam.

You are **not** allowed to:

- Have any kind of communication during the exam.
- Use any "calculator" tool that claims to solve logic problems automatically. (These things are not reliable: there are many, many equivalent ways to present logics, and each such tool is specific to one presentation. We will notice if you use a tool like this.)
- Try to look up answers using Google or other search engines. The exam is designed to make this hard, but please don't test it.

This has been a very unusual and complicated period. To avoid making it even more complicated, we are putting a lot of trust in you with this take home exam. Please do not abuse our trust!

Write your answers on separate sheets of paper. You will submit each answer separately on Canvas. Do not submit your scratch work. Make your solutions stand out clearly.

Happy proving!

Grade Table (for teacher use only)

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	0	13	25	10	8	8	16	10	10	100
Score:										

1. (0 points) **IMPORTANT**: your exam will not be graded if you do not submit this.

   The VU has released the following statement of integrity for online exams.

   > I hereby declare, that I understand that taking an online exam during this corona crisis is an emergency measure to prevent study delays as much as possible. I know that fraud control will be tightened and I also realize that a special appeal is now being made to trust my integrity. With this statement, I promise to:
   >
   > - make this exam completely on my own,
   > - do not consult sources, other than explicitly mentioned in the examination,
   > - make myself available for any oral explanation of my answers.

   Please write the following statement and sign your name below it.

   > I have read the statement of integrity and exam rules, and followed them while taking this exam.

2. (13 points) This task concerns validity and (un)satisfiability in propositional and predicate logic.

   For this task, the answers suffice. No justification is necessary.

   Determine for each of the following formulas whether it is valid, satisfiable (but not valid), or unsatisfiable.

   (a) (2 points) $(q \rightarrow p) \vee (p \rightarrow q)$

   (b) (2 points) $\forall x \forall y (R(x, y) \rightarrow R(y, x))$

   (c) (2 points) $p \wedge \neg q \wedge (q \rightarrow p)$

   Determine for each of the following sets of formulas whether the set is consistent or inconsistent:

   (d) (2 points) $\{ \ \forall x (P(x) \vee Q(x)),\ \exists x \neg P(x),\ \exists x \neg Q(x) \ \}$

   (e) (2 points) $\{ \ \forall x \exists y R(x, y),\ \forall x \exists y \neg R(x, y) \ \}$

   (f) (3 points) Write down a formula in propositional logic that is satisfiable but not valid. (Don't use any of the formulas in parts (a)-(c) above.)

3. (25 points) This task concerns natural deduction and countermodels in propositional and first order logic.

   Determine for each of the given semantic implications whether it is valid.

   - For the valid implications, give a derivation using natural deduction.
   - For the invalid implications, give a counter model. (In the case of propositional logic, give a truth assignment of the variables that shows the implication is invalid.)

   (a) (5 points) $a \vee (b \to c)) \models (\neg a \wedge b) \to c$

   (b) (5 points) $\forall x \neg R(x,x) \models \forall x \exists y \neg R(x,y)$

   (c) (5 points) $\models (p \vee q \vee r) \to ((p \wedge q) \vee (p \wedge r) \vee (q \wedge r))$

   (d) (5 points) $\models \forall x \exists y R(x,y) \to \exists x \exists y (R(x,y) \wedge R(y,x))$

   (e) (5 points) The natural deduction proof below contains two mistakes: proof rules have been applied incorrectly. Identify the mistaken lines, and explain in your own words how you would correct the proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x P(x)}\,(3) \qquad \dfrac{\dfrac{\dfrac{}{\forall x(P(x) \to Q(x))}\,(2)}{P(a) \to Q(a)}\,\forall E \qquad \dfrac{}{P(a)}\,(4)}{Q(a)}\,\to E}{\exists x Q(x)}\,\exists E\ (4)}{\exists x P(x) \to \exists x Q(x)}\,\to I\ (3)}{(P(a) \to Q(a)) \to \exists x P(x) \to \exists x Q(x)}\,\to I\ (2)}{\forall x(P(x) \to Q(x)) \to \exists x P(x) \to \exists x Q(x)}\,\forall I\ (1)$$

4. (10 points) This task concerns the translation of natural language sentences to predicate logic.

   The domain is a group of people. The meaning of the symbols is:

a : Alma	$K(x, y)$: x knows y
g : Gustav	$L(x, y)$: x loves y
	$H(x, y)$: x was husband of y

   Translate the following sentences into formulas of predicate logic:

   (a) (2 points) Everybody who knows Gustav also knows Alma.

   (b) (2 points) Not all of Almas lovers' lovers love Alma.

   (c) (3 points) Alma had two (or more) husbands.

   (d) (3 points) Apart from Gustav, Alma had precisely two other husbands.

5. (8 points) This task concerns meta-theorems of predicate logic, and expressibility of properties in predicate logic.

   (a) (4 points) Explain the structure of the argument for why model finiteness is not expressible by a formula in predicate logic. (A proof sketch is sufficient. It may contain "then it can be shown that..." in places. The basic argument should be clear, as should which statements are used and which theorems are applied.)

   (b) (4 points) State the soundness theorem of predicate logic. Explain how we could use this theorem to show that there is no natural deduction proof (without hypotheses) of the formula $\exists x P(x) \to \forall x P(x)$.

6. (8 points) This task concerns decision problems and their solvability (decidability).

   (a) (3 points) Explain, *in your own words*, what it means for a decision problem to be decidable.

   (b) (5 points) The *termination problem*, checking whether a program terminates with a certain input, is undecidable. I've tried to prove this below, but I made a mistake!

   Identify the flaw in my argument, and fix it: give a good argument that the termination problem is undecidable. Your argument should have around the same amount of detail as my bad attempt.

   Theorem. The termination problem is undecidable.

   Proof. Assume there is a program T_{self} that decides the termination problem.

   T_{self} takes as input a program P, and outputs yes if P terminates on input P, and no otherwise.

   Define a new program Z. Z takes as input a program P, and uses T_{self} to test if P terminates on input P. If yes, Z loops forever. If no, Z terminates.

   What happens if we run Z on input Z?

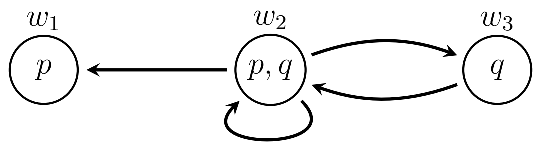
   - If Z terminates on input Z, then the definition of Z says that it does not terminate.
   - If Z does not terminate on input Z, then the definition of Z says that it does terminate.

   So Z terminates on input Z if and only if it doesn't terminate on input Z. A contradiction!

   Therefore no program can decide the termination problem.

7. (16 points) This task concerns basic modal logic and Kripke structures.

Given is the Kripke model $\mathcal{M} = (W, R, L)$ as drawn in the following figure:



Determine *for each* world x whether or not the following statements hold. You do not have to explain your answer.

(a) (3 points) $\mathcal{M}, x \models \Diamond p \to p$

(b) (3 points) $\mathcal{M}, x \models \Box\Box q$

We now consider the underlying frame $\mathcal{F} = (W, R)$. Determine whether the following statements hold. If yes, give an argument why. If no, give a counterexample.

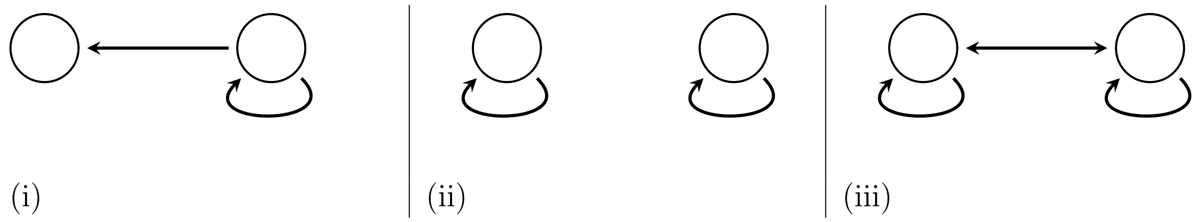
(c) (3 points) $\Diamond p \to \Diamond\Diamond p$

(d) (3 points) $(p \wedge q) \to \Diamond q$

Draw a Kripke model with three worlds, where the following sentence is true at two of the worlds and false at the third. Clearly identify at which worlds the sentence is true.

(e) (4 points) $\Box\Diamond(p \wedge q)$

8. (10 points) This task is about correspondence of formulas and frame properties in modal logic.



(a) (3 points) I have a Kripke model $\mathcal{M} = (W, R, L)$. I tell you that $M \models p \vee \Diamond\top$. Which of the frames (i)–(iii) above could be the underlying frame $\mathcal{F} = (W, R)$? (There could be more than one.)

Briefly explain your answer.

(b) (3 points) I've changed my mind: now I claim that $\mathcal{F} \models p \vee \Diamond\top$. Which of the frames (i)–(iii) above could be $\mathcal{F}$ now? (There could be more than one.)

Again, briefly explain your answer.

(c) (4 points) The formula $\Box p \to \Diamond p$ does *not* characterize the frame property of functionality (the property that every world has exactly one successor).

Show this by providing a counterexample, and give a short explanation of why your counterexample works.

9. (10 points) This task is about binary relations.

   (a) (3 points) Consider a set of elements $U = \{a, b, c\}$. Define a strict partial order $<$ on this set. That is: for each pair of elements $x, y \in U$, write down whether or not $x < y$.

   (b) (3 points) Is the order that you wrote down in part (a) a total order on U? Why or why not?

   (c) (4 points) Give an example of a binary relation on the natural numbers $\mathcal{N}$ that is reflexive, symmetric, and transitive, but is not the equality relation. For example: the $\leq$ relation would *not* be a good answer, because it is not symmetric. The relation $R(x, y)$ that holds when x and y are both even or both odd is a good answer.