2
a) valid
b) sat, not valid
c) sat, not valid
d) consistent
e) consistent
f) $p \land q$

3 a) valid

$$\dfrac{\dfrac{a \lor (b \supset c) \qquad \dfrac{\dfrac{\overline{a}^{\,2} \qquad \dfrac{\overline{\neg a \land b}^{\,1}}{\neg a}}{\bot}}{c} \qquad \dfrac{\overline{b \supset c}^{\,3} \qquad \dfrac{\overline{\neg a \land b}^{\,1}}{b}}{c}}{c}\;{\scriptstyle 2,3}}{(\neg a \land b) \to c}\;{\scriptstyle 1}$$

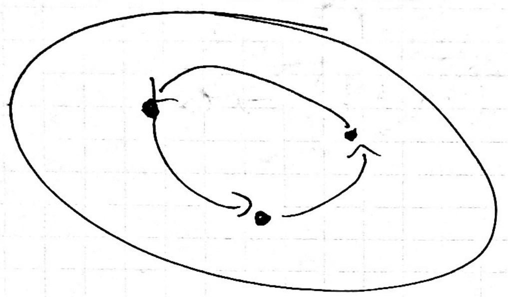
b) valid

$$\dfrac{\dfrac{\dfrac{\forall x\, \neg R(x,x)}{\neg R(x,x)}}{\exists y\, \neg R(x,y)}}{\forall x\, \exists y\, \neg R(x,y)}$$

3 c) invalid

$p \to$ true, $q \to$ false, $r \to$ false

d) invalid



e) The first step (1) should have been $\to I$, since the main connective is $\to$.

Step (4) is a bad application of $\exists E$. Instead of $Q(a)$, the new goal should be $\exists x Q(x)$ as before.

(You could also point out these two lines and prove it correctly)

4. a) $\forall x(K(x,g) \to K(x,a))$

b) $\neg \forall x(\underbrace{\exists y(L(x,y) \wedge L(y,a))}_{x \text{ is a lover of a lover of Alma}} \to L(x,a))$

y is a lover of Alma

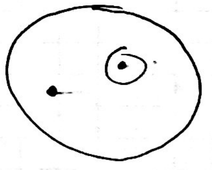
c) $\exists x \exists y(H(x,a) \wedge H(y,a) \wedge x \neq y)$

d) $\exists x \exists y(H(x,a) \wedge H(y,a) \wedge x \neq y \wedge x \neq g \wedge y \neq g \wedge \forall z(H(z,a) \Rightarrow z=g \vee z=x \vee z=y)))$

5a) in slides

b) Soundness: For any formula φ of first order logic, if $\vdash \varphi$ (φ is derivable) then $\models \varphi$ (φ is semantically valid)

$\exists x P(x) \to \forall x P(x)$ is not valid. Here's a countermodel:



P points are circled ⊙

So if I could derive this sentence, the soundness theorem says it must be true in this model. But it isn't. So I can't derive it.

7. a) w_1: yes

w_2: yes

w_3: no

b) w_1: yes

w_2: no

w_3: no

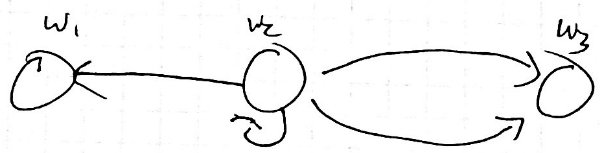
c) It holds. Suppose we have a labeling L on F.
$M = \langle W, R, L \rangle$

$M, w_1 \models \Diamond p \Rightarrow \Diamond\Diamond p$ trivially, since $M, w_1 \not\models \Diamond p$ no matter what the labeling is.

Suppose $M, w_2 \models \Diamond p$. Then $M, w_2 \models \Diamond\Diamond p$, since $R(w_2, w_2)$ holds.

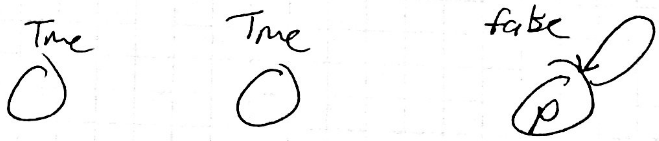
Suppose $M, w_3 \models \Diamond p$. Then $M, w_2 \models p$, since w_2 is the only world accessible from w_3. So $M, w_2 \models \Diamond p$, and thus $M, w_3 \models \Diamond\Diamond p$.

d) It does not hold. Here is a countermodel.



$w_2 \not\models \Box q$, since $w_1 \not\models q$. Thus $w_2 \not\models \Box\Box q$.

e.



8. a) i, ii, iii could all be possible.

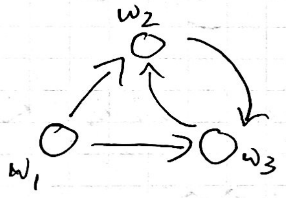
If I choose a labeling where p is true at every world, then $M \models p \vee \Diamond\top$ in each model.

b) ii, iii could be $\mathcal{F}$. $\Diamond\top$ holds at every world under every labeling.

i cannot work. A labeling that makes p false at the left world is a counterexample.

c) A counterexample is the frame

$\mathcal{F} =$



$\mathcal{F} \models \Box p \to \Diamond p$. Pick any labeling and world w, suppose $M, w \models \Box p$. Since w must see some world w' (every world sees another), $M, w' \models p$. So $M, w \models \Diamond p$.

But $\mathcal{F}$ is not functional: w_1 has two outgoing arrows.

9. a) $a<b$ ~~another~~
$b<c$
$a<c$
Nothing else: $\neg(a<a)$, $\neg(c<a)$, etc.

b) Yes. For any $x, y \in U$, either $x<y$, $y<x$, or $x=y$.

c) $R(x,y) \Leftrightarrow$ x and y are both even or x and y are both odd
Reflexive: x has the same parity as x
Symmetric: if x and y have the same parity, then y and x do
transitive: if x and y have the same parity and y and z do, then x and z do