Final Exam
Logic and Modelling
January 7, 2016

General information:

- The exam consists of 4 pages and 8 tasks.
- The maximal number of points is 100.
- The grade for this exam is (points + 11)/11.

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1. This task concerns validity, and (un)satisfiability in propositional and predicate logic.

Tell for each of the formulas whether it is valid (tautology), satisfiable or a contradiction:
(the answer suffices, no explanation is needed)

(a) $((a \to b) \to a) \to a$ **(2 points)**

(b) $(a \vee b) \wedge (\neg a \vee b) \wedge \neg b$ **(2 points)**

(c) $(a \to b) \vee (\neg b \to \neg a)$ **(2 points)**

Answer the following:

(d) What does it mean that a set of formulas of predicate logic is **syntactically** consistent? **(2 points)**

Tell for each of the sets whether they are consistent:
(the answer suffices, no explanation is needed)

(e) $\{\ \exists x\, \forall y\, R(x,y),\ \exists y\, \forall x\, \neg R(x,y)\ \}$ **(2 points)**

(f) $\{\ \forall x\, (\ \neg S(k,x) \to S(x,x)\ ),\ \forall x\, \neg S(x,x)\ \}$ **(2 points)**

(g) $\{\ \forall x\, (\ P(x) \vee Q(x)\ ),\ \exists x\, \neg P(x),\ \forall x\, \neg Q(x)\ \}$ **(2 points)**

2. This task concerns natural deduction and counter models.

Determine for each of the given semantic implications whether it is valid.

- For the valid implications, give a derivation using natural deduction.
- For the invalid implications, give a counter model (in case of propositional logic give a truth assignment of the variables that shows that the implication is wrong).

(a) $a \to b,\ a \to \neg b \models \neg a$ **(6 points)**

(b) $\forall x\, R(x,x) \models \forall y\, \exists x\, R(x,y)$ **(6 points)**

(c) $\forall x\, \exists y\, \exists z\, \big(R(x,y) \wedge R(y,z) \wedge R(z,x) \big) \models \exists x\, \exists y\, \big(R(x,y) \wedge R(y,x) \big)$ **(6 points)**

(d) $\exists x\, \forall y\, R(x,y) \models \forall y\, \exists x\, R(x,y)$ **(6 points)**

3. This task concerns the translation of sentences in natural language to predicate logic (with or without equality).

The domain is a group of people. The meaning of the symbols is:

a :	Alma	$K(x,y)$:	x knows y
g :	Gustav	$L(x,y)$:	x loves y
		$H(x,y)$:	x was husband of y

Translate the following sentences into formulas of predicate logic (with or without equality).

(a) Everybody is known by somebody. **(3 points)**

(b) Everybody who knows Alma also loves Alma. **(3 points)**

(c) Not all of Alma's lovers' lovers love her. **(3 points)**

(d) Alma had precisely two husbands. **(4 points)**

4. Determine the truth value of the sentence ϕ defined as:

$$\forall y\ \exists x\ \mathit{loves}(x, y)$$

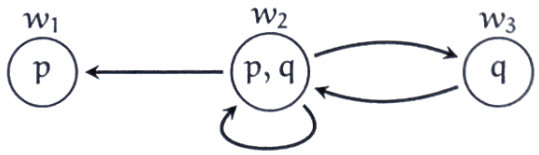
in the $\langle \mathit{alma}/0, \mathit{loves}/2 \rangle$-model:

$$\mathcal{M} \begin{cases} A = \{a, g, f\} \\ \mathit{alma}^{\mathcal{M}} := a \\ \mathit{loves}^{\mathcal{M}} := \{\langle g, a\rangle, \langle f, a\rangle, \langle a, g\rangle\} \end{cases}$$

Argue in *formal steps* on the basis of the truth definition for formulas in models (in the stepwise manner shown in the lectures). **(3 points)**

5. This task concerns basic modal logic and Kripke structures.

Given is the Kripke model $\mathcal{M} = (W, R, L)$ as drawn in the following figure:



Determine for each world x whether or not the following holds:
(the answer suffices, no explanation is needed)

(a) $\mathcal{M}, x \Vdash \Box p \to q$ Not **(3 points)**

(b) $\mathcal{M}, x \Vdash \Box\Diamond q$ hold **(3 points)**

We now consider the underlying frame $\mathcal{F} = (W, R)$. Determine whether:
(the answer suffices, no explanation is needed)

(c) $\mathcal{F} \models \Diamond p \to \Box(p \vee q)$ H **(2 points)**

(d) $\mathcal{F} \models \Diamond\Diamond p \to \Diamond\Diamond\Diamond p$ Not **(2 points)**

The following task is about correspondence of formulas with frame properties:

(e) Show that for every frame $\mathcal{F}$ that is not transitive we have: $\mathcal{F} \not\models \Diamond\Diamond p \to \Diamond p$.
Please give a detailed argumentation, as has been used in the lectures. **(4 points)**

6. This task concerns meta-theorems of predicate logic, and expressibility of properties in predicate logic.

(a) Formulate the soundness and completeness theorem of predicate logic in terms of the (symbols for the) syntactical and semantical entailment relations. Make clear the difference between soundness and completeness. Paraphrase this theorem verbally in a sentence. **(4 points)**

(b) Formulate the compactness theorem of predicate logic. **(3 points)**

(c) Exhibit, for $n = 1$, $n = 2$, and schematically for general n, formulas ψ_n in predicate logic with equality such that, for all $n \in \mathbb{N}$, $n \geq 1$, the formula ψ_n is true in a model $\mathcal{M}$ if and only the domain of $\mathcal{M}$ contains at most n values. **(3 points)**

(d) State the consistency theorem of predicate logic, and prove it from the soundness and completeness theorem. **(5 points)**

7. This task concerns decision problems and their solvability (decidability).

(a) When is a decision problem undecidable (unsolvable)? **(2 points)**

(b) Give an instance of Post's Correspondence Problem (PCP) that has a no solution. **(2 points)**

(c) Formulate and explain the undecidability theorem for formula validity in predicate logic. **(2 points)**

(d) Is the decision problem for entailment statements $\Gamma \vdash \phi$, where Γ is a finite set of formulas of predicate logic and ϕ a formula of predicate logic, decidable (solvable)? Justify your answer. **(4 points)**

8. This task concerns program logic.

We want a terminating program P with the property that the output value (after execution) of x is the input value (before execution) of y plus 2, and the output value of y is the input value of x times 2.

(a) Formulate this program specification for P as an appropriate satisfiability statement concerning a Hoare triple. **(3 points)**

(b) This task is about the proof system for partial correctness.
Write down and explain the composition rule. **(4 points)**