Linear Algebra for AI
Resit Exam 2023
05-06-2023, 18:45–22:00

Name: ______________________________

Student ID: ______________________________

Question:	1	2	3	4	5	6	7	Total
Points:	7	7	5	7	5	3	2	36
Score:								

Grade: ______

$$\text{Grade} = \frac{\#\text{total points}}{4} + 1$$

Read the instructions carefully!

- The use of a calculator, book or notes is **not** allowed.
- Answer the questions in the spaces provided. If you run out of room for an answer, continue on the back of the page, but make sure the final answer starts on the front.
- You can ask for extra paper to do your calculations, but you shall not submit it, the graded answers will be the ones on these papers, so make sure the final answers are properly explained here, starting in the blank space provided and finishing on the back if necessary.
- Answers without supporting work will not be given full credit, so make sure to explain the theoretical basis of what you do for full score.
- If you feel like you will be short in time, prioritize questions you know or the ones with higher score count.

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1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a map with standard matrix

$$A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}.$$

(a) **(1 pts.)** Is the map linear?

(b) **(3 pts.)** For which values of a is the map invertible?

(c) **(3 pts.)** For the values for which T is not invertible, find $\ker T$.
Hint: Provide a basis for $\ker T = \{x \in \mathbb{R}^3 : T(x) = 0\}$.

2. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ -2 & 2 & 3 \end{bmatrix}$.

(a) **(2 pts.)** Determine the LU-factorization of A.

(b) **(2 pts.)** Let $b = \begin{bmatrix} 4 \\ 11 \\ 1 \end{bmatrix}$. Use the LU factorization to solve the system $Ax = b$.

(c) **(3 pts.)** Find the inverse of A.

3. Let V be the set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in \mathbb{R}$ and $a + b + c + d = 0$.

(a) **(3 pts.)** Show that V is a subspace of the vector space of all 2×2 matrices with real entries.

(b) **(2 pts.)** Let W be the subset of V in which $a \leq d$. Is W also a vector subspace? Explain your answer.

4. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}.$$

(a) **(1 pts.)** Show that $\mathbf{v_1} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of A and find its corresponding eigenvalue λ_1.

(b) **(4 pts.)** Determine the remaining eigenvalues of A and give a basis for every eigenspace of A.

(c) **(2 pts.)** Determine whether A is diagonalizable. If so, give the diagonalization, i.e., an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why A is not diagonalizable.

5. Consider the following basis for $\mathbb{R}^3$: $\left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) **(4 pts.)** Using the given basis, find an orthonormal basis for $\mathbb{R}^3$.

(b) **(1 pts.)** Calculate the distance between $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$.

6. Mark each of the following statements as true or false. If the statement is true, give a proof. If the statement is false, give a proof or provide a counterexample.

   (a) **(1 pts.)** An orthogonal set cannot be linearly dependent.

   (b) **(1 pts.)** If A is a matrix whose columns form a basis $\mathcal{A}$ of a vector space and $\mathbf{x}$ is a coordinate vector in the standard basis, then $[\mathbf{x}]_{\mathcal{A}} = A\mathbf{x}$.

   (c) **(1 pts.)** Let A be a square matrix. If A is similar to A^2, then $\lambda = 0$ is an eigenvalue of A.

7. **(2 pts.)** Let

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 4 \end{bmatrix},$$

$$A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 2 \\ 1 & 1 & 1 & \cdots & 0 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & n-1 \\ 1 & 1 & 1 & \cdots & 1 & n \end{bmatrix}$$

with $n > 1$. Show that $\det A_n = 1$, for all $n > 1$.
Note: Do not use induction on n, but rather calculate the determinant of A_n.
Hint: Make use of the properties of the determinant of a matrix on row operations.