Linear Algebra for AI
Resit Exam 2023

① Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a map with standard matrix $A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}$.

a) Is the map linear?

Yes because it is a matrix map, and a map given by a matrix is linear.

b) For which values of a is the map invertible?

We know that a matrix map is invertible if and only if its standard matrix is invertible. So that's what we have to check. In this case we'll use determinants:

$\det A = a\begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} + \begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix} = a(a^2-1)-(a-1)+1-a = a^3-a-a+1+1-a = a^3-3a+2=0 \longrightarrow a=1$ is a root, so

$a^3-3a+2 = \underbrace{(a^2+a-2)}(a-1)$

$a = \frac{-1 \pm \sqrt{1+4\cdot 2}}{2} = \frac{-1 \pm 3}{2}$

$a=1$
$a=-2$

So $a=1$ and $a=-2$ make $\det A = 0$ and hence the map not invertible. So $a \in \mathbb{R} \setminus \{1,-2\}$ will make it invertible.

c) For the values for which T is not invertible, find Ker T.

Finding Ker T for matrix maps is the same as finding Nul A. Let's start with $a=1$, we want to find all solutions to $Ax=0$.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow x_1 = -x_2 - x_3 \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, so $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for Ker T.

For $a=-2$, $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \left.\begin{matrix} x_1 = x_3 \\ x_2 = x_3 \end{matrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so a basis for Ker T is $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

② Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ -2 & 2 & 3 \end{bmatrix}$.

a) Determine the LU-factorization of A.

We can do so by row reducing A: $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ -2 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = U$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$

b) Let $b = \begin{bmatrix} 4 \\ 11 \\ 1 \end{bmatrix}$. Use the LU-factorization to solve the system $Ax=b$.

Since $A = LU$, $Ax=b \longrightarrow LUx=b \longrightarrow L(Ux)=b$, so take $y=Ux$ and start by solving $Ly=b$: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 1 \end{bmatrix}$, by augmenting and row-reducing:

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 2 & 1 & 0 & 11 \\ -1 & 3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 3 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right]$, so $y = \begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix}$, which means $Ux=y$, by the same method: $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 4 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim$
$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$, so $x = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$.

c) Find the inverse of A.

We know that if such inverse exists, then $[A \,\vdots\, I] \sim [I \,\vdots\, A^{-1}]$. Let's do that:

$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 2 & 0 & 1 & 0 \\ -2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 3 & 4 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 4 & 7 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7/4 & -3/4 & 1/4 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & -3/4 & 3/4 & -1/4 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7/4 & -3/4 & 1/4 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 5/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7/4 & -3/4 & 1/4 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/8 & -1/8 & -1/8 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7/4 & -3/4 & 1/4 \end{array}\right]$, so $A^{-1} = \begin{bmatrix} 5/8 & -1/8 & -1/8 \\ -2 & 1 & 0 \\ 7/4 & -3/4 & 1/4 \end{bmatrix}$.

③ Let V be the set of matrices as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a+b+c+d=0$, $a,b,c,d \in \mathbb{R}$.

a) Show that V is a subspace.

For that we show it satisfies the three conditions:

(i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$ since $0+0+0+0=0$.

(ii) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in V$, is the sum in V? $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$, and $a+a'+b+b'+c+c'+d+d' = (a+b+c+d)+(a'+b'+c'+d') = 0+0=0$, so yes.

(iii) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ and $s \in \mathbb{R}$, is the scalar product in V? $s\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}$, and $sa+sb+sc+sd = s(a+b+c+d) = s0 = 0$, so yes.

b) Let $W \subset V$ such that $a \leq d$. Is W also a subspace?

No, since the closure under multiplication is not satisfied. Take for instance $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in V$, but $-1\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \notin V$ since $1 \nleq -1$.

④ Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$.

a) Show that $v_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector and find its eigenvalue λ_1.

We have to show that $Av_1 = \lambda_1 v_1$, so $Av_1 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, so $\lambda_1 = -1$.

b) Determine the remaining eigenvalues of A and give a basis for each eigenspace.

We find the remaining eigenvalues by solving the characteristic equation $\det(A-\lambda I)=0$. $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 1-\lambda & -2 \\ -1 & 0 & -2-\lambda \end{vmatrix} = -1\begin{vmatrix} 1 & 1 \\ 1-\lambda & -2 \end{vmatrix} + (-2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} =$
$= -(-2-1+\lambda) + (-2-\lambda)((2-\lambda)(1-\lambda)-2) = -\lambda+3+(-2-\lambda)(\cancel{2}-2\lambda-\lambda+\lambda^2-\cancel{2}) = -\lambda+3+(-2-\lambda)(\lambda^2-3\lambda) = -\lambda+3-2\lambda^2+6\lambda-\lambda^3+3\lambda^2 = -\lambda^3+\lambda^2+5\lambda+3=0$. We know $\lambda_1=-1$ is a solution so we can divide our characteristic polynomial by $(\lambda+1)$ to get $-\lambda^2+2\lambda+3$, whose roots are $\lambda = \frac{-2 \pm \sqrt{4+4\cdot 3}}{-2} = \frac{-2\pm 4}{-2} = \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 3 \end{cases}$
which means $\lambda_1=-1$ has multiplicity 2. So the eigenvalues are $\lambda_1=-1$ and $\lambda_2=3$.

Let's find a basis for their eigenspaces by solving $(A-\lambda I)v=0$. Start with $\lambda_1=-1$.

$A+I = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & -2 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & -8 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \nearrow \left.\begin{matrix} x_1+x_3=0 \\ x_2-2x_3=0 \end{matrix}\right\} \left.\begin{matrix} x_1=-x_3 \\ x_2=2x_3 \end{matrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, so a basis is $\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\}$.

Now for $\lambda_2=3$.

$A-3I = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & -2 \\ -1 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix} \nearrow \left.\begin{matrix} x_1+5x_3=0 \\ x_2+6x_3=0 \end{matrix}\right\} \left.\begin{matrix} x_1=-5x_3 \\ x_2=-6x_3 \end{matrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -5 \\ -6 \\ 1 \end{bmatrix}$, so a basis is $\left\{\begin{bmatrix} -5 \\ -6 \\ 1 \end{bmatrix}\right\}$.

c) Is A diagonalizable?

No, because $\lambda_1=-1$ has multiplicity 2 as eigenvalue but its eigenspace has dimension 1. We can also say that the sum of dimensions of the eigenspaces is 2, not 3, so because of that it is not diagonalizable.

⑤ Consider the following basis $\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\}$.

a) Find an orthonormal basis.

Since our basis is not orthogonal we apply Gram-Schmidt to orthogonalize it first:

$u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$.

$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1/2}{3/2}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/6 \\ -1/6 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$. Now that we have an orthogonal basis, we have to normalize. So first we calculate the modulus of the vectors, $\|u_1\| = \sqrt{2}$, $\|u_2\| = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$, $\|u_3\| = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$, now $w_i = \frac{1}{\|u_i\|}u_i$. So $w_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$, $w_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$.
So $\{w_1, w_2, w_3\}$ is our orthonormal basis.

b) Calculate the distance between $y = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\text{Span}\{v_1, v_2\}$.

Notice that $y = v_1 + v_2$, which means that $y \in \text{Span}\{v_1, v_2\}$ and therefore $\text{dist}(y, \text{Span}\{v_1, v_2\}) = 0$. Nonetheless, if this is to be checked using orthogonal projections, the basis for the spanned space is required to be orthogonal. Since $\text{Span}\{v_1, v_2\} = \text{Span}\{u_1, u_2\}$ from a) we can calculate the orthogonal projection of y, which is y itself, so the distance vector is $y - y = 0$ anyway.

6 True or false?

a) An orthogonal set cannot be linearly dependent.

It's hard to say just true or false. Strictly speaking it's false because $\{\vec{0}\}$ is orthogonal and linearly dependent. If we assume the set to not be the zero vector, then it's true by a theorem that states that every orthogonal set of nonzero vectors is linearly independent.

b) If the columns of A form a basis $\mathcal{A}$, then $[x]_{\mathcal{A}} = Ax$.

False. It's $[x]_{\mathcal{A}} = A^{-1}x$, or $x = A[x]_{\mathcal{A}}$.

c) $A_{n\times n}$. If A is similar to A^2, then $\lambda = 0$ is an eigenvalue of A.

If A is similar to A^2, then there exists an invertible matrix P such that $A = PA^2P^{-1}$. If we look at the determinant, $\det A = \det(PA^2P^{-1}) = \det P \cdot \det A^2 \cdot \det P^{-1} = \det P \cdot \det P^{-1} \cdot \det A^2 = \det A^2$, so we have $\det A = \det A^2 = \det(A\,A) = \det A \cdot \det A = (\det A)^2$, so $\det A = (\det A)^2 \longrightarrow (\det A)^2 - \det A = 0 \rightarrow \det A(\det A - 1) = 0$ which means either $\det A = 0$ or $\det A = 1$, and if $\det A = 0$ then $\lambda = 0$ would be an eigenvalue, but since it can happen that $\det A = 1$, in that case $\lambda = 0$ would not be an eigenvalue, so false.

7 Let $A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 2 \\ 1 & 1 & 1 & \cdots & 0 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & n-1 \\ 1 & 1 & 1 & \cdots & 1 & n \end{bmatrix}$, with $n > 1$. Show that $\det A_n = 1$, for all $n > 1$.

We know that adding a multiple of one row onto another does not change the determinant. Thus if we substract the first row from the second we obtain $\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 0 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & n-1 \\ 1 & 1 & 1 & \cdots & 1 & n \end{bmatrix}$.

If we substract it now from the third we get $\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 1 & \cdots & 0 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & n-1 \\ 1 & 1 & 1 & \cdots & 1 & n \end{bmatrix}$ which still needs one more reduction step.
Also notice how the 3 in the last column turned to a 2 after one step of reduction.
If we now substract the second from the third we get $\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & n-1 \\ 1 & 1 & 1 & \cdots & 1 & n \end{bmatrix}$, now in reduced form the 2 turned into a 1. So for each row i we will need $i-1$ reduction steps, which will always bring the last column to a 1. After reducing for all rows, we get the following triangular matrix $\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$ for which the determinant is the product of its diagonal, so 1.