Contents

Chapter 1

Eigenvalues and eigenvectors

1.1 Eigenvalues and eigenvectors

Definition 1 *An* eigenvector *of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. λ is called a* eigenvalue *of A if there is a nonotrivial solution $\vec{x}$ of $A\vec{x} = \lambda\vec{x}$.*

An eigenvalue may be zero, but a eigenvector must always be nonzero. Every eigenvalue has at least one corresponding eigenvector. If 0 is an eigenvalue of A, then A is not invertible.

It is easy to check whether a certain vector is an eigenvector. For example: check if $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$.

$$A\vec{v} = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 2\vec{v} \tag{1.1}$$

And therefore an eigenvector.

It is also easy to check if a given λ is an eigenvalue. For example: check if 1 is an eigenvalue of $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$. Since $A\vec{x} = \lambda\vec{x}$ must hold for it to be an eigenvalue. It must also be true that

$$(A - \lambda I)\vec{x} = \vec{0} \tag{1.2}$$

In our example that gives us

$$A - 1I = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \tag{1.3}$$

These columns are linearly dependent, and therefore $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution and thus is it in eigenvalue.

To find the corresponding eigenvector row reduce to get

$$\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \tag{1.4}$$

And thus all vectors of the form $x_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ with $x_2 \neq 0$ are eigenvectors.

The subspace of $\mathbb{R}^n$ that contains all the eigenvectors corresponding to a certain eigenvalue is called the *eigenspace* corresponding to λ.

When given a matrix finding the eigenvalues is often not easy. In one case however it is.

Theorem 1 *The eigenvalues of a triangular matrix are the entires on its main diagonal.*

Theorem 2 *If $\vec{v_1}, \ldots, \vec{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, then $\vec{v_1}, \ldots, \vec{v_r}$ are linearly independent.*

1.1.1 Exercises

Exercise 1

Is $\lambda = -2$ an eigenvalue of $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$?

Exercise 2

Is $\lambda = 3$ an eigenvalue of $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

Exercise 3

Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue.

Exercise 4

Find a basis for the eigenspace by $\lambda = -2$ and $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$.

1.2 Characteristic equation

1.2.1 Finding eigenvalues

To have be an eigenvalue the equation $(A - \lambda I)\vec{x} = \vec{0}$ must have a nontrivial solution. Which is the same as saying $\det(A - \lambda I) = 0$. This is called the *characteristic equation* of A. This gives us a polynomial in λ and by solving that we can find the eigenvalues.

For example: $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$. $A - \lambda I = \begin{bmatrix} -2-\lambda & 12 \\ -1 & 5-\lambda \end{bmatrix}$.

$$\begin{aligned} det\ A = (-2-\lambda)(5-\lambda) + 12 = 0 \qquad (1.5) \\ \lambda^2 - 3\lambda + 2 = 0 \\ (\lambda - 1)(\lambda - 2) = 0 \\ \lambda_1 = 1 \text{ and } \lambda_2 = 2 \end{aligned}$$

The number of times a certain λ occurs in the characteristic equation is called the *algebraic multiplicity* of λ.

1.2.2 Exercises

Exercise 5

Use the characteristic polynomial to find the eigenvalues. $A = \begin{bmatrix} 3 & -6 \\ 1 & -4 \end{bmatrix}$.

Exercise 6

Use cofactor expansion to find the eigenvalues. $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

1.3 Diagonalization

1.3.1 Similarity

Two matrices, A and B, are *similar* if there exists a invertible matrix P, such that $A = PBP^{-1}$.

Theorem 3 *If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues.*

But be warned. If two matrices have the same eigenvalues, they are not necessarilly similar.

1.3.2 Diagonalization

The concept of similarity combined with a diagonal matrix is very usefull when calculating powers of matrices because $A^k = PDP^{-1}PDP^{-1}\ldots PDP^{-1} = PD(P^{-1}P)D(P^{-1}\ldots P)DP^{-1} = PDID\ldots IDP^{-1} = PD^kP$. With D diagonal and therefore very easy to calculate.

A matrix is called *diagonalizable* if it can be written as $A = PDP^{-1}$.

Theorem 4 *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond to the eigenvectors in P.*

To diagonalize a certain matrix we have to follow these steps:

1. Find the eignevalues of A.
2. Find three linearly independent eigenvectors of A.
3. Construct P from the vectors in step 2.
4. Construc D from the corresponding eigenvalues.

For example: diagonalize $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ calculations with the characteristic equation gives us the following eigenvalues and corresponding eigenvectors. $\lambda_1 = 0$ with $\vec{v_1} = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}$, $\lambda_2 = -4$ with $\vec{v_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\lambda_3 = 3$ with $\vec{v_3} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$ These eigenvectors are linearly independent thus we can construct

$$P = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 6 & 2 & 3 \\ -13 & 1 & -2 \end{bmatrix} \tag{1.6}$$

And

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{1.7}$$

Finally P^{-1} can also be calculated and then A is diagonalized.

1.3.3 Determining whether a matrix is diagonizable

Theorem 5 *An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.*

Theorem 6 *Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.*

1. *For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.*
2. *The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if*
   (a) *The characteristic polynomial factors completely into linear factors and*
   (b) *The dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.*
3. *If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $B_1, \ldots, B_p$ forms an eigenvector basis for $\mathbb{R}^n$.*

1.3.4 Exercises

Exercise 7

Diagonalize $A = \begin{bmatrix} -7 & 2 \\ 13 & 4 \end{bmatrix}$.

Exercise 8

Diagonalize $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$ which eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$.

1.4 Linear transformations

Let V be an n-dimensional vectorspace and W an m-dimensional vectorspace, with bases B and C respectively, and T a linear transformation from V to W. The *matrix for T relative to the bases B and C M* is given by

$$M = \begin{bmatrix} [T(\vec{b_1})]_C & [T(\vec{b_2})]_C & \dots & [T(\vec{b_n})]_C \end{bmatrix} \tag{1.8}$$

If $V = W$ then $[T\vec{x}]_B = [T]_B[\vec{x}]_B$ for any $\vec{x}$ in V.
$[T]_B$ is called the *B-matrix for T*.

Theorem 7 *Suppose $A = PDP^{-1}$, where D is diagonal $n \times n$. If B is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the B-matrix for the transformation $\vec{x} \mapsto A\vec{x}$.*

1.4.1 Exercises

Exercise 9

Let C and B be bases for V and W respectively and let $T : V \to W$ and $T(\vec{c_1}) = 3\vec{b_1} - \vec{b_2}$, $T(\vec{c_2}) = -\vec{b_1} - 2\vec{b_2}$. Find the matrix for T relative to C and B.

Exercise 10

Let E be the standard basis for $\mathbb{R}^3$, B a basis for a vector space V and $T : \mathbb{R}^3 \to V$, whith the property $T(x_1, x_2, x_3) = (x_3 - x_2)\vec{b_1} - (x_1 + x_3)\vec{b_2} + (x_1 - x_2)\vec{b_3}$. Find the matrix for T relative to E and B.

Exercise 11

Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be a transformation that maps a polynomial $\vec{p}(t)$ into the polynomial $\vec{p}(t) + t^2\vec{p}(t)$. Find the matrix for T relative to the standard bases for $\mathbb{P}_2$ and $\mathbb{P}_4$.

1.5 Complex eigenvalues and eigenvectors

It can be that when solving a characteristic polynomial you get a complex solution. This gives us a *complex eigenvalue* and a *complex eigenvector* in $\mathbb{C}^n$.

The complex conjugate of a $n \times n$ matrix A with complex numbers, denoted as $\overline{A}$, is formed by taking the complex conjugate of every entrie.
$Re\ \vec{x}$ and $Im\ \vec{x}$ are the real and imaginary parts of a $\vec{x}$ in $\mathbb{C}^n$ respectivelly.

If λ is a complex eigenvalue of A, then $\overline{\lambda}$ is also a complex eigenvalue of A. The same holds for $\vec{v}$ and $\overline{\vec{v}}$.

If $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ a and b both real and not both zero. Then the eigenvalues of C are $\lambda = a \pm bi$. Also let r be the modulus of λ, that is $r = |\lambda|$. Then

$$C = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} cos\ \phi & -sin\ \phi \\ sin\ \phi & cos\ \phi \end{bmatrix} \tag{1.9}$$

With ϕ the argument of λ.

Theorem 8 *Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ (b not zero) and an associated eigenvector $\vec{v}$ in $\mathbb{C}^2$. Then*

$$A = PCP^{-1}, \text{ where } P = \begin{bmatrix} Re\ \vec{v} & Im\ \vec{v} \end{bmatrix} \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{1.10}$$

1.5.1 Exercises

Exercise 12

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$.

Exercise 13

$A = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$ list the eigenvalues, the angle and the rotation of $\vec{x} \mapsto A\vec{x}$.

Exercise 14

$A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$. Find P and C such that $A = PCP^{-1}$.

1.6 Application to discrete dynamical systems

In this section we study $\vec{x_{k+1}} = A\vec{x_k}$, with $\vec{v_1}, \ldots, \vec{v_n}$ linearly independent eigenvectors of A and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, ordered from biggest to smallest. The eigenvectors form a basis for $\mathbb{R}^n$, thus every $\vec{x_0}$ can be written as

$$\vec{x_0} = c_1\vec{v_1} + \ldots + c_n\vec{v_n} \tag{1.11}$$

Then $\vec{x_k}$ can be calculated by

$$\vec{x_k} = c_1(\lambda_1)^k\vec{v_1} + \ldots + c_n(\lambda_n)^k\vec{v_1} \tag{1.12}$$

When A is 2×2 the graph of $\vec{x_0}, \vec{x_1}, \ldots$ is called the *trajectory*. If both eigenvalues are smaller then 1 the origin is a *attractor*. If they are both greater then 1, then the origin is a *repeller*. The origin is a *saddle point* if one eigenvalue is greater then 1 an the other smaller then 1.

To find the lines along which the points go, we need a diagonal matrix. If A is already diagonal the points go along the axis. If A is not diagonal, diagonalize A and the points go along the eigenvectors. The direction of greatest attraction or repultion is along the eigenvector corresponding to the smallest or greatest eigenvalue respectively.

If A has complex eigenvectors the points will spiral outwards if the absolute value of the eigenvalues is greater then 1 and will spiral towards the origin if the eigenvectors are smaller then 1.

1.6.1 Exercises

Exercise 15

Suppose A is 3×3, with eigenvalues 3, 4/5 and 3/5 and corresponding eigenvalues $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$ and $\vec{v_3} = \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$. Let $\vec{x_0} = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. Find the solution of the equation $\vec{x_{k+1}} = A\vec{x_k}$ for the specified $\vec{x_0}$ and describe what happens when $k \to \infty$.

Exercise 16

Is the origin a attractor, repeller or saddle point? Find the directions of greatest attraction and/or repulsion. $A = \begin{bmatrix} 0.3 & 0.4 \\ -0.3 & 1.1 \end{bmatrix}$.

Chapter 2

Least squares

2.1 Innerproduct

2.1.1 Innerproduct

The *innerproduct* in $\mathbb{R}^n$ of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u} \cdot \vec{v}$, is defined as $\vec{u}^T \vec{v}$.

For example: $\vec{u} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 9 \end{bmatrix}$, caluculate $\vec{u} \cdot \vec{v} = \begin{bmatrix} 2 & -5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 9 \end{bmatrix} = 2(-1) + (-5)2 + 6 * 9 = 42$.

Theorem 9

1. *$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.*
2. *$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.*
3. *$(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$.*
4. *$\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$.*

2.1.2 Length

Definition 2 *The* length *or* norm *of $\vec{v}$ is defined by $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.*

By this definition $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$. And $\|c\vec{v}\| = |c|\|\vec{v}\|$.

A vector with length 1 is called a *unit vector*. Multiplying a vector with 1 divided by its lenght makes it a unit vector. This is called *normalizing*.

2.1.3 Distance

Definition 3 *The* distance *between $\vec{u}$ and $\vec{v}$ is $dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$.*

2.1.4 Orthogonality

Definition 4 *Two vectors $\vec{u}$ and $\vec{v}$ are* orthogonal *if $\vec{u} \cdot \vec{v} = 0$.*

Theorem 10 *Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.*

2.1.5 Orthogonal complement

The orthogonal complement of W subspace of $\mathbb{R}^n$, $W^\perp$, is the set of all vectors that are orthogonal to all vectors that span W.

Theorem 11 *Let A be an $n \times m$ matrix, then:* $(Row\ A)^\perp = Nul\ A$ *and* $(Col\ A)^\perp = Nul\ A^T$.

2.1.6 Angles

For vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ there exists a formula with the angle between these two vectors. $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| cos\phi$.

2.1.7 Exercises

Exercise 17

$\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}$. What is $\vec{u} \cdot \vec{v}$? What is distance between $\vec{u}$ and $\vec{v}$? Normalize $\vec{u}$ and $\vec{v}$. Are $\vec{u}$ and $\vec{v}$ orthogonal to each other?

2.2 Orthogonal sets

2.2.1 Orthogonal sets

A *orthogonal set* in $\mathbb{R}^n$ is a set whose vectors are all orthogonal to each other. That is $\vec{u_i} \cdot \vec{u_j} = 0$ if $i \neq j$.

Theorem 12 *If S is a orthogonal set, then are its vectors linearly independent and thus a basis for the subspace spanned by S.*

2.2.2 Orthogonal basis

Definition 5 *A* orthogonal basis *for a subspace W is a basis for W that is also a orthogonal set.*

As the next theorem shows is that a big advantage of orthogonal bases is that there is no need to row reduce matrices to find the weights for a linear combination.

Theorem 13 *The weights in the linear combination $\vec{y} = c_1\vec{u_1} + \ldots + c_p\vec{u_p}$ are given by $c_j = \frac{\vec{y}\cdot\vec{u_j}}{\vec{u_j}\cdot\vec{u_j}}$ For each $\vec{y}$ in W and $\{\vec{u_1}, \ldots, \vec{u_p}\}$ orthogonal basis for W.*

2.2.3 Orthonormal sets

A *orthonormal set* is a orthogonal set of unit vectors. And is a *orthonormal basis* for the subspace spanned by the vectors in the set.

Theorem 14 *An $m \times n$ matrix U has orthonormal columns if and only if $U^TU = I$.*

Theorem 15 *If U is a $m \times n$ matrix with orthonormal columns then,*

1. $\|U\vec{x}\| = \|\vec{x}\|$.
2. $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$.
3. $(U\vec{x}) \cdot (U\vec{y}) = 0$ *if and only if* $\vec{x} \cdot \vec{y} = 0$.

It is said that the mapping $\vec{x} \mapsto U\vec{x}$ preserves length and orthogonality.

If a square matrix has orthonormal columns then it is called a *orthogonal matrix*. For such a matrix U holds that $U^{-1} = U^T$.

2.3 Orthogonal projections

2.3.1 In $\mathbb{R}^2$

Given a vector $\vec{u}$ in $\mathbb{R}^2$ consider the problem of decomposing a other vector $\vec{y}$ into the sum of two vectors $\vec{y} = \vec{\hat{y}} + \vec{z}$ with $\vec{\hat{y}}$, the *the orthogonal projection $\vec{y}$ onto $\vec{u}$*, a multiple of $\vec{u}$ and $\vec{z}$ orthogonal to $\vec{u}$. If L is the line through $\vec{u}$, then

$$\vec{\hat{y}} = proj_L \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \tag{2.1}$$

2.3.2 In $\mathbb{R}^n$

To extend the idea of orthogonal projections to $\mathbb{R}^n$ we need to realize that a vector in $\mathbb{R}^n$ with a orthogonal basis, always can be written as the sum of a vector in the span of a number of vectors from the basis and a vector in the span of the other vectors in that basis.

Theorem 16 *Let W be subspace of $\mathbb{R}^n$ then every $\vec{y}$ in $\mathbb{R}^n$ can be written uniquely in the form $\vec{y} = \vec{\hat{y}} + \vec{z}$. Where $\vec{\hat{y}}$ is in W and $\vec{z}$ is in $W^\perp$. If $\{\vec{u_1}, \ldots, \vec{u_p}\}$ is any orthogonal basis of W then,*

$$proj_W \vec{y} = \vec{\hat{y}} = \frac{\vec{y} \cdot \vec{u_1}}{\vec{u_1} \cdot \vec{u_1}} \vec{u_1} + \ldots + \frac{\vec{y} \cdot \vec{u_p}}{\vec{u_p} \cdot \vec{u_p}} \vec{u_p} \tag{2.2}$$

2.3.3 Properties

If $\vec{y}$ is in W, then $proj_W \vec{y} = \vec{y}$.

Theorem 17 $\|\vec{y} - \vec{\hat{y}}\| < \|\vec{y} - \vec{v}\|$ *, for all $\vec{v}$ in W distinct from $\vec{\hat{y}}$. Thus $\vec{\hat{y}}$ is the point in W closest to $\vec{y}$.*

Theorem 18 *If $\{\vec{u_1}, \ldots, \vec{u_p}\}$ is a orthonormal basis for W, then $proj_W \vec{y} = (\vec{y} \cdot \vec{u_1})\vec{u_1} + \ldots + (\vec{y} \cdot \vec{u_p})\vec{u_p}$. If $U = \begin{bmatrix} \vec{u_1} & \ldots & \vec{u_p} \end{bmatrix}$ then $proj_W \vec{y} = UU^T \vec{y}$.*

2.3.4 Exercises

Exercise 18

$\{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$ is a orthogonal basis. $\vec{u_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{u_3} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$ and $\vec{u_4} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$. Write $\vec{x} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$ as the sum of two vectors, one in $Span\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ and one in $Span\{\vec{u_4}\}$.

Exercise 19

Find the closest point to $\vec{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$ in the subspace W spanned by $\vec{u_1} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{u_2} = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$. And calucate the distance between $\vec{y}$ and W.

2.4 Making a orthogonal basis

2.4.1 Gram-Schmidt process

Given a certain basis for a supspace you can make a orthogonal basis for that subspace via the following process.

Theorem 19 *Given a basis* $\{\vec{x_1}, \ldots, \vec{x_p}\}$ *then if*

$$\vec{v_1} = \vec{x_1} \tag{2.3}$$

$$\vec{v_2} = \vec{x_2} - \frac{\vec{x_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} \tag{2.4}$$

$$\vec{v_3} = \vec{x_3} - \frac{\vec{x_3} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} - \frac{\vec{x_3} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2} \tag{2.5}$$

$$\vec{v_p} = \vec{x_p} - \frac{\vec{x_p} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} - \frac{\vec{x_p} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2} - \ldots - \frac{\vec{x_p} \cdot \vec{v_p}}{\vec{v_p} \cdot \vec{v_p}} \vec{v_p} \tag{2.6}$$

$\{\vec{v_1}, \ldots, \vec{v_p}\}$ *is a orthogonal basisi for Span* $\{\vec{x_1}, \ldots, \vec{x_p}\}$.

For example find a orthogonal basis for the column space of matrix $A = \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix}$.

$\vec{v_1} = \vec{a_1} = \begin{bmatrix} 12 \\ 6 \\ -4 \end{bmatrix}$, $\vec{v_2} = \vec{a_2} - \frac{\vec{a_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} = \begin{bmatrix} -51 \\ 167 \\ 24 \end{bmatrix} - \frac{294}{196} \begin{bmatrix} 12 \\ 6 \\ -4 \end{bmatrix} = \begin{bmatrix} -69 \\ 158 \\ 30 \end{bmatrix}$, $\vec{v_3} = \vec{a_3} - \frac{\vec{a_3} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} - \frac{\vec{a_3} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2} = \begin{bmatrix} 4 \\ -68 \\ -41 \end{bmatrix} - \frac{-196}{196} \begin{bmatrix} 12 \\ 6 \\ -4 \end{bmatrix} - \frac{-12250}{30625} \begin{bmatrix} -69 \\ 158 \\ 30 \end{bmatrix} = \begin{bmatrix} -58/5 \\ 6/5 \\ -33 \end{bmatrix}$.

By normalizing these vectors you can make a orthonormal basis.

2.4.2 QR factorization

Theorem 20 *A $m \times n$ matrix A can be written as $A = QR$ with Q an $m \times n$ matrix whose columns form a orthonormal basis for $Col\ A$ and R $n \times n$ upper triangular invertible with positive entries on its diagonal.*

Q can be constructed by taking the columns that form a basis for $Col\ A$ and applying the Gram-Schmidt process to get a orthogonal basis and normalizing these vectors. Because Q is orthonormal $Q^T Q = I$, thus if $A = QR$, then $R = Q^T A$.

For example: $A = \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix}$. From the example in subsection 2.4.1 we have orthogonal vectors, normalizing these gives us $\vec{q_1} = \begin{bmatrix} 6/7 \\ 3/7 \\ -2/7 \end{bmatrix}$, $\vec{q_2} = \begin{bmatrix} -69/175 \\ 158/175 \\ 6/35 \end{bmatrix}$ and $\vec{q_3} = \begin{bmatrix} -58/175 \\ 6/175 \\ -33/35 \end{bmatrix}$ placing these vectors in a matrix gives us $Q = \begin{bmatrix} 6/7 & -69/175 & -58/175 \\ 3/7 & 158/175 & 6/175 \\ -2/7 & 6/35 & -33/35 \end{bmatrix}$. Then $R = Q^T A = \begin{bmatrix} 14 & 21 & -14 \\ 0 & 175 & -70 \\ 0 & 0 & 35 \end{bmatrix}$.

2.4.3 Exercises

Exercise 20

Find an orthogonal basis for the subspace W spanned by $\vec{x_1} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{x_2} = \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix}$.

Exercise 21

Find an orthogonal basis for the column space of $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$.

Exercise 22

Find the QR decomposition of $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$.

2.5 Least-squares problems

When $A\vec{x} = \vec{b}$ has no solution we can calculate a good approximation.

Definition 6 *A* least-squares solution *of $A\vec{x} = \vec{b}$ is an $\vec{\hat{x}}$ such that $\|\vec{b} - A\vec{\hat{x}}\| \leq \|\vec{b} - A\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$.*

Theorem 21 *The set of least-squares solutions of $A\vec{x} = \vec{b}$ coincides with the nonempty set of solutions of the normal equations $A^T A\vec{x} = A^T\vec{b}$.*

Theorem 22 *Let A be $m \times n$. The following statements are logically equivalent.*

1. *The equation $A\vec{x} = \vec{b}$ has a unique least-squares solution for each $\vec{b}$ in $\mathbb{R}^m$.*
2. *The columns of A are linearly independent.*
3. *The matrix $A^T A$ is invertible.*

If these are true, then $\vec{\hat{x}} = (A^T A)^{-1} A^T \vec{b}$.

The *least-squares error* of an approximation is $\|\vec{b} - A\vec{\hat{x}}\|$.

For example $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$. We need to calculate $A^T A\vec{\hat{x}} = A^T\vec{b} \rightarrow \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} \rightarrow \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$ Then by either row reducing or by calculating the inverse you can find that $\vec{\hat{x}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The least square error then is $\|\vec{b} - A\vec{\hat{x}}\| = \| \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} \| = (-2)^2 + (-4)^2 + 8^2 = 84$.

2.5.1 Alternative calculations

When A has orthogonal columns then $\hat{x_p}$ is given by $\frac{\vec{b} \cdot \vec{a_p}}{\vec{a_p} \cdot \vec{a_p}}$.

Theorem 23 *If $A = QR$ then $\vec{\hat{x}} = R^{-1} Q^T \vec{b}$.*

Instead of calculating $\vec{\hat{x}} = R^{-1} Q^T \vec{b}$ it is often easyer to row reduce $R\vec{\hat{x}} = Q^T \vec{b}$.

2.5.2 Exercises

Exercise 23

Calculate the least squares solution of $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$. And give the least squares error.

Exercise 24

Calculate the least squares solution of $A\vec{x} = \vec{v}$ with $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

2.6 Least-squares in linear models

Given a set of data (X_i, Y_i), we may wish to draw a line, $Y_i = \beta_0 + \beta_1 X_i$, through these points.

In a perfect sitiuation we have

$$\begin{aligned} Y_1 &= \beta_0 + \beta_1 X_1 \\ Y_2 &= \beta_0 + \beta_1 X_2 \\ &\vdots \\ Y_n &= \beta_0 + \beta_1 X_n \end{aligned} \tag{2.7}$$

Wich can also be written as

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \tag{2.8}$$

Or as

$$\vec{Y} = X\vec{\beta} \tag{2.9}$$

If the points are not on a perfect line we can find β_0 and β_1 that makes the line the best by calculating the least-squares solution of this system, thus by finding the solution of $X^T\vec{Y} = X^T X$.

In some cases just a normal line does not fit the data right. Then in general we can find $y = \beta_0 f_0(x) + \cdots + \beta_n f_n(x)$ by placing the $f_k(x)$ in X and the β_k in $\vec{\beta}$. For example we can find a parabola with $\vec{Y} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ & \vdots & \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$

2.6.1 Exercises

Exercise 25

Find the least squares line $y = \beta_0 + \beta_1 x$ that fits best the data points $(2, 3)$, $(3, 2)$, $(5, 1)$ and $(6, 0)$.

2.7 Inner product spaces

2.7.1 Inner product

Definition 7 *An* inner product *on a vector space V is a function $\langle \vec{u}, \vec{v} \rangle$ that gives a real number and satisfies*

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$.
2. $\langle u \vec{+} v, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$.
3. $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$.
4. $\langle \vec{u}, \vec{u} \rangle \geq 0$ *and* $\langle \vec{u}, \vec{u} \rangle = 0$ *if and only if* $\vec{u} = \vec{0}$.

A vector space with an inner product is a inner product space*.*

For polynomials the following inner product is often used. Let $t_0, \cdots, t_n$ be distinct nummers. For p and p in $\mathbb{P}_n$ define $\langle p, q \rangle = p(t_0)q(t_0) + \cdots + p(t_n)q(t_n)$. Checking each of the axioms will show that this is indeed a inner product.

With this we can for example calculate $\langle p, q \rangle$ if $t_0 = -1,\ t_1 = 2,\ t_3 = 3,\ p(t) = 6 - t$ and $q(t) = t + t^2$. $\langle p, q \rangle = p(-1)q(-1) + p(2)q(2) + p(3)q(3) = 7 * 0 + 4 * 6 + 3 * 12 = 60$.

An other inner product is the one for $C[a, b]$. Defined as $\langle p, q \rangle = \frac{1}{b-a} \int_a^b p(t)q(t)dt$.

2.7.2 Length, distance and orthogonality

The *length* of a vector is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$. The *distance* between $\vec{u}$ and $\vec{v}$ is $\|\vec{v} - \vec{u}\|$. Two vectors are *orthogonal* if $\langle \vec{u}, \vec{v} \rangle = 0$.

2.7.3 Gram-Schmidt and projections

The Gram-Schmidt process can not only be applied to $\mathbb{R}^n$ but also to other inner product spaces to create orthogonal bases. The same holds for projections.

For example the inner product defined for polynomials evaluated at -2, -1, 0, 1 and 2. And the polynomials 1, t and t^2. The vector of values for 1, t and t^2 are respectivelly $\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$, $\begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}$ and $\begin{bmatrix}4\\1\\0\\1\\4\end{bmatrix}$. Note that $\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix} = 0$ hence $\langle 1, t \rangle = 0$. Therefore the first two vectors of the basis are 1 and t. The last vector can be created by $t^2 - \frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle t^2, t \rangle}{\langle t, t \rangle} t = t^2 - 2$.

2.7.4 Best approximation

Also best approximations using projections can be calculated using the inner product.

2.7.5 Cauchy-Schwartz and Triangle inequallity

Theorem 24 $|\langle\vec{u},\vec{v}\rangle| \leq \|\vec{u}\|\|\vec{v}\|$.

Theorem 25 $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.

2.7.6 Exercises

Exercise 26

For the given inner product for polynomials evaluated at the points -2, -1 and 0, calculate $\langle p, q\rangle$ with $p(t) = t - 3t^2$ and $q(t) = 4 + 2t$. Also calculate $\|p\|$ and $\|q\|$. Lastly calculate the orthognal projection of q on the subspace spanned by p.

Exercise 27

Let $V = C[-1, 1]$ and let the inner product be calculated as in this section. $f(t) = t + t^2$ and $g(t) = 1 - 3t$. Calculate $\langle f, g\rangle$.

Chapter 3

Symmetric matrices and the SVD

3.1 Symmetric matrices

3.1.1 Orthogonal diagonalizations

A *symmetric matrix* is a matrix such that $A^T = A$.

Theorem 26 *If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.*

If in the diagonlization $A = PDP^{-1}$ P is an orthogonal matrix, then A is said to be *orthogonal diagonizable.* And $P^{-1} = P^T$.

Theorem 27 *An $n \times n$ matrix is orthogonally diagonizable if and only if it is a symmetric matrix.*

It is however not enough to put all the eigenvectors in a matrix. When there is a eigenspace with dimension 2, then Gramm-Schmidt must be applied to get the second orthogonal eigenvector.

For example: $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ which eigenvalues and corresponding eigenspaces are -1: $\vec{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. And 5: $\vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The first two eigenvectors are not orthogonal, thus we apply Gram-Schmidt. $\vec{z_2} = \vec{v_2} - \frac{\vec{v_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$. Normalizing and placing the vectors $\vec{v_1}$, $\vec{z_2}$ and $\vec{v_3}$ gives us P.

3.1.2 Spectral theorem and decomposition

Theorem 28 *An symmetric $n \times n$ matrix A has the following properties:*

1. *A has n real eigenvalues, counting multiplicities.*
2. *The dimension of each eigenspace equals the multiplicity of λ.*

If $A = PDP^{-1} = \begin{bmatrix} \vec{u_1} & \cdots & \vec{u_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u_1}^T \\ \vdots \\ \vec{u_n}^T \end{bmatrix}$, then the *spectral decomposition* of A is $A = \lambda_1 \vec{u_1}\vec{u_1}^T + \cdots + \lambda_n \vec{u_n}\vec{u_n}^T$.

3.1.3 Exercises

Exercise 28

Orthogonal diagonalize $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{bmatrix}$. The eigenvalues are -4, 4 and 7.

Exercise 29

Orthogonalize $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. The eigenvalues are -2 and 7.

3.2 Quadratic forms

A *quadratic form* is a function such that $Q(\vec{x}) = \vec{x}^T A\vec{x}$. When calculating quadratic forms you will notice that there is a link with symmetric matrices. For example if $A = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}$ then its quadratic form is $2x_1^2 - 2x_1x_2 - 2x_1x_2 - 3x_2^2 = 2x_1^2 - 4x_1x_2 + x_2^2$. Or if $Q(\vec{x}) = 4x_1^2 + x_2^2 - 2x_3^2 + 3x_1x_2 + 12x_2x_3$ then A in the quadratic form is $A = \begin{bmatrix} 4 & 3/2 & 0 \\ 3/2 & 1 & 6 \\ 0 & 6 & -2 \end{bmatrix}$.

3.2.1 Change of variable

Preferably we want to loose the cross term. In order to do that we orthogonal diagonalize $A = PDP^{-1}$. Then the new quadratic form is $\vec{y}^T D\vec{y} = (P^{-1}\vec{x})^T D(P^{-1}\vec{x})$.

3.2.2 Classifying quadriatic forms

Definition 8 *A qaudratic form Q is:*

1. *Positive definite if $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$.*
2. *Negative definite if $Q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$.*
3. *Indefinte if both negative and positive values are attained.*

Theorem 29 *Let A be $n \times n$ symmetric, then its quadratic form is:*

1. *positive definite if and only if the eigenvalues of A are all positive.*
2. *negative definite if and only if the eigenvalues of A are all negative.*
3. *indefinite if and only if A has both positive and negative eigenvalues.*

3.2.3 Exercises

Exercise 30

Calculate the quadratic form $\vec{x}^T A\vec{x}$ with $\vec{x} = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Exercise 31

Find the matrix of the quadratic form: $5x_1^2 + 2x_3^2 - 6x_1x_2 + x_2x_3$.

Exercise 32

Consider $3x_1^2 + 6x_2^2 - 4x_1x_2$. Classify the quadratic form. Also make a change of variable to get a quadratic form with no cross-product term.

3.3 Constrained optimalization

Sometimes we want to maximize or minimize a quadratic form under the condition that $\|\vec{x}\| = 1$.

Theorem 30 *If A is a symmetric matrix, then its maximum, under the codition that$\|\vec{x}\| = 1$, is the greates eigenvalue of A and is attained when $\vec{x}$ is the unit eigenvector of A corresponding to that eigenvalue. The minimum is equal to the smallest eigenvector and is attained when $\vec{x}$ is the eigenvector corresponding to that eigenvalue.*

Sometimes we say we have aditional constraints, for example that the $\vec{x}$ is orthogonal with the eigenvector corresponding to the largest eigenvalue. Then the solution of the optimalization is the second largest eigenvalue and its eigenvector. If it also has to be orthogonal to the second largest, then take the third eigenvalue. Etc.

However, the constraint does not always gives us a problem with unit vectors. In that case we need to rewrite the constraint such that it is about unitvectors.

3.3.1 Exercises

Exercise 33

Find the maximum value of $Q(\vec{x})$ subject to the constraint $\vec{x}^T\vec{x} = 1$, find the vector $\vec{u}$ where this maximum is attained and find the maximum of $Q(\vec{x})$ subject to the constraints $\vec{x}^T\vec{x} = 1$ and $\vec{x}^T\vec{u} = 0$. $Q(\vec{x}) = 3x_1^2 + 3x_2^2 + 5x_3^2 + 6x_1x_2 + 2x_1x_3 + 2x_2x_3$.

3.4 Singular value decomposition

3.4.1 Singular values

Let A be a $m \times n$ matrix. Then $A^T A$ is a symmetric matrix and if $\lambda_1, \cdots, \lambda_n$ are its eigenvalues sorted from greatest to smallest. The *singular values* of A are $\sigma_i = \sqrt{\lambda_i}$. Which is the same as the lengths of $A\vec{v_i}$ if $\vec{v_i}$ is a eigenvalue of $A^T A$.

Theorem 31 *The set $A\vec{v_1}, \cdots, A\vec{v_r}$ with $\vec{v_i}$ eigenvalues of $A^T A$ corresponding to nonzero singular values, is a orthogonal basis for Col A and rank$A = r$.*

3.4.2 Singular value decomposition

Let $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ an $m \times n$ matrix with D $r \times r$ diagonal.

Theorem 32 *Let A be a $m \times n$ matrix with rank r. Then there exist an $m \times n$ matrix Σ which its D contains the first r singular values of A. And there exist a $m \times m$ orhtogonal matrix U and an $n \times n$ orthogonal matrix V such that $A = U\Sigma V^T$.*

V can be constructed by placing the eigenvectors of $A^T A$ sorted in a matrix. U by placing $\frac{1}{\sigma_i} A\vec{v_i}$ for $i = 1, \cdots, r$ together with $m - r$ other vectors that make up a orthonormal basis for $\mathbb{R}^m$ in a matrix.

3.4.3 Finding a SVD composition

Step 1 is finding the eigenvalues and corresponding orthonormal eigenvectors of $A^T A$. Step 2 is constructing Σ and V. Step 3 is constructing U.

For example $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$ then $A^T A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$ which eigenvalues and corresponding eigenvectors are $\lambda_1 = 18$: $\vec{v_1} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and $\lambda_2 = 0$: $\vec{v_2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Thus $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. $\sigma_1 = \sqrt{18}$ and $\sigma_2 = 0$, therefore $\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. To construct U calculate $A\vec{v_1} = \begin{bmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix}$ and $A\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. $\vec{u_1} = \frac{1}{\sigma_1} A\vec{v_1} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$. The other columns of U must be chosen is such a way that the columns of U form a orthonormal basis for $\mathbb{R}^3$. Thus both other vectors must satisfy $\vec{u_1}^T \vec{x} = 0$, which is the same as $x_1 - 2x_2 + 2x_3 = 0$. That gives us $\vec{w_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{w_2} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Applying Gram-Schmidt with normalizations gives $\vec{u_2} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$ and $\vec{u_3} = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$. Finally construct U and write $A = U\Sigma V^T$.

3.4.4 Exercises

Exercise 34

Find the singular values decomposition of $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$.

Chapter 4

Answers

Answer of exercise 1

$A - \lambda I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$. The equation $A - \lambda I = \vec{0}$ has a nonzero solution, therefore $\lambda = -2$ is an eigenvalue. The general solution of $(A - \lambda I)\vec{x} = \vec{0}$ is $x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$. Thus an eigenvalue is $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$.

Answer of exercise 2

$A - \lambda I = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$. Augmenting with a colulmn of zeros and row reducing gives

$$\begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $A - \lambda I = \vec{0}$ has a nonzero solution, therefore $\lambda = 3$ is an eigenvalue. The general solution of $(A - \lambda I)\vec{x} = \vec{0}$ is $x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Thus an eigenvalue is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Answer of exercise 3

Calculate $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Thus it is an eigenvector with eigenvalue -2.

Answer of exercise 4

$A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix}$. Augmenting with a column of zeros and row reducing gives us $\begin{bmatrix} 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & -13 & 3 & 0 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus the general solution is $x_3 \begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$ therefore a basis for the eigenspace is $\begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$.

Answer of exercise 5

$det(A - \lambda I) = det(\begin{bmatrix} 3 & -6 \\ 1 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}) = \begin{vmatrix} 3-\lambda & -6 \\ 1 & -4-\lambda \end{vmatrix} = (3-\lambda)(-4-\lambda) - 12 + 6 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0$, $\lambda_1 = -3$, $\lambda_2 = 2$.

Answer of exercise 6

$det(A - \lambda I) = det(\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda)(4-\lambda) = 0$, $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 4$.

Answer of exercise 7

First we have to find the eigenvalues and eigenvectors. $det(A - \lambda I) = \begin{vmatrix} -7-\lambda & 2 \\ 13 & 4-\lambda \end{vmatrix} = (-7-\lambda)(4-\lambda) - 26 = \lambda^2 + 3\lambda - 54 = (\lambda - 6)(\lambda + 9) = 0$, $\lambda_1 = 6$, $\lambda_2 = -9$. Eigenvector corresponding to λ_1: $\begin{bmatrix} -7-6 & 2 & 0 \\ 13 & 4-6 & 0 \end{bmatrix} \sim \begin{bmatrix} -13 & 2 & 0 \\ 13 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} -13 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $x_1 = 2/13 x_2$, therfore $\vec{v_1} = \begin{bmatrix} 2/13 \\ 1 \end{bmatrix}$. Eigenvector corresponding to λ_2: $\begin{bmatrix} -7+9 & 2 & 0 \\ 13 & 4+9 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 0 \\ 13 & 13 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $x_1 = -x_2$, theefore $\vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Now we can construct PDP^{-1}. $P = \begin{bmatrix} 2/13 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & -9 \end{bmatrix}$, $P^{-1} = (1/(13/15) \begin{bmatrix} 1 & 1 \\ -1 & 2/13 \end{bmatrix} = \begin{bmatrix} 13/15 & 13/15 \\ -13/15 & 26/195 \end{bmatrix}$. Optionally you could have scaled the eigenvectors to simplify calculations.

Answer of exercise 8

We need to find the eigenvectors. Eigenvectors corresponding to λ_1: $\begin{bmatrix} -2 & -4 & -6 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which gives us $\vec{v_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ which are clearly linearly independent. Eigenvector corresponding λ_2: $\begin{bmatrix} -1 & -4 & -6 & 0 \\ -1 & -1 & -3 & 0 \\ 1 & 2 & 4 & 0 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which gives us $\vec{v_3} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$. Now we can construct PDP^{-1}. $P = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, inverting P

yields $P^{-1} = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 2 & 4 \\ -1 & -2 & -3 \end{bmatrix}$.

Answer of exercise 9

$[T(\vec{c_1})]_B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $[T(\vec{c_2})]_B = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, therefore $M = \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix}$.

Answer of exercise 10

First of all compute $T(\vec{e_1}) = 0\vec{b_1} - \vec{b_2} + \vec{b_3}$, $T(\vec{e_2}) = -\vec{b_1} + 0\vec{b_2} - \vec{b_3}$ and $T(\vec{e_3}) = \vec{b_1} + \vec{b_2} + 0\vec{b_3}$. Then calculate $[T(\vec{e_1})]_B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $[T(\vec{e_2})]_B = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$ and $[T(\vec{e_3})]_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. And thus $M = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

Answer of exercise 11

Compute $T(1) = 1 + t^2$, $T(t) = t + t^3$, $T(t^2) = t^2 + t^4$. Expressed in the standard basis $T(1) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $T(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $T(t^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. Thus $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Answer of exercise 12

$det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 3 = \lambda^2 - 4\lambda + 7 = 0$, $\lambda_1 = \frac{1}{2}(4 - \sqrt{4^2 - 4 * 1 * 7}) = 2 - \frac{1}{2}\sqrt{-12} = 2 - i\sqrt{3}$ then $\lambda_2 = \overline{\lambda_1} = 2 + i\sqrt{3}$. The eigenvalue corresponding to λ_1: $\begin{bmatrix} 2-(2-i\sqrt{3} & -3 & 0 \\ 1 & 2-(2-i\sqrt{3} & 0 \end{bmatrix}$ Because we know the first and second row must be multiples of each other we can set the second row to all zeros. $\sim \begin{bmatrix} i\sqrt{3} & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $i\sqrt{3}x_1 = 3x_2$. Now pick $x_1 = 1$, $i\sqrt{3} = 3x_2$, $x_2 = 1/3i\sqrt{3}$ therefore $\vec{v_1} = \begin{bmatrix} 1 \\ 1/3i\sqrt{3} \end{bmatrix}$ and $\vec{v_2} = \overline{\vec{v_1}} = \begin{bmatrix} 1 \\ -1/3i\sqrt{3} \end{bmatrix}$.

Answer of exercise 13

Because $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ we can easily see that the eigenvalues are $\sqrt{3} \pm i$. $r = |\lambda| = \sqrt{a^2 + b^2} = \sqrt{3+1} = 2$. $\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} cos\ \phi & -sin\ \phi \\ sin\ \phi & cos\ \phi \end{bmatrix}$ So $2cos\phi = \sqrt{3}$, $\phi = \frac{\pi}{6}$.

Answer of exercise 14

First find the eigenvalues and eigenvectors of A. $\begin{vmatrix} 3-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda)+2 = \lambda^2-7\lambda+14 = 0$, $\lambda = 1/2(7\pm\sqrt{-7}) = 7/2\pm(1/2)i\sqrt{7}$. Now find the eigenvectors $\begin{bmatrix} 3-(7/2-1/2\sqrt{7}) & -1 \\ 2 & 4 \end{bmatrix}$, $(-1/2+1/2i\sqrt{7})x_1 = x_2$, $x_1 = 1$, $x_2 = -1/2+1/2i\sqrt{7}$, $\vec{v_1} = \begin{bmatrix} 1 \\ -1/2+1/2i\sqrt{7} \end{bmatrix}$. Now we can construct P and C. $P = \begin{bmatrix} 1 & 0 \\ -1/2 & 1/2\sqrt{7} \end{bmatrix}$ and $C = \begin{bmatrix} 7/2 & -1/2\sqrt{7} \\ 1/2\sqrt{7} & 7/2 \end{bmatrix}$.

Answer of exercise 15

To use the given formula we need to solve $\vec{x_0} = c_1\vec{v_1} + \ldots + c_n\vec{v_n}$ in this case that will give us the matrix $\begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ -3 & -5 & 7 & 3 \end{bmatrix} \ldots \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ then the formula gives us $\vec{x_k} = c_1(\lambda_1)^k\vec{v_1} + \ldots + c_n(\lambda_n)^k\vec{v_1} = \vec{x_k} = 2(3)^k\vec{v_1} + 1(4/5)^k\vec{v_2} + 2(3/5)^k\vec{v_3}$. When $k \to \infty$ $\vec{x_k}$ will look more and more like $2(3)^k\vec{v_1}$ because the other parts tend to zero.

Answer of exercise 16

We need to find the eigenvalues $\begin{vmatrix} 0.3-\lambda & 0.4 \\ -0.3 & 1.1-\lambda \end{vmatrix} = \lambda^2 - 1.4\lambda + 0.45$. Using the quadratic formula we find $\lambda_1 = 0.5$ and $\lambda_2 = 0.9$. Both eigenvalues are smaller than 1 so the origin is a attractor. For the direction of greatest attraction we need to find the eigenvector corresponding to λ_1. $\begin{bmatrix} -0.2 & 0.4 & 0 \\ -0.3 & 0.6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\vec{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The direction of greatest attraction is along that eigenvector.

Answer of exercise 17

$\vec{u}\cdot\vec{v} = 3(-2)+(-1)3+6*2 = 3$. The distance between $\vec{u}$ and $\vec{v} = \|\vec{u}-\vec{v}\| = \| \begin{bmatrix} 5 \\ -4 \\ -4 \end{bmatrix} \| = \sqrt{57}$.

To normalize the vectors we need to multiply both vectors with 1 divided by their length, so first calculate $\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{14}$ and $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{49} = 7$. Then the normalized vectors are $\vec{u}' = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{v}' = \frac{1}{7} \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}$. $\vec{u}$ and $\vec{v}$ are not orthogonal to each other because their inproduct is not equal to zero.

Answer of exercise 18

We need to write $\vec{x} = \hat{\vec{x}} + \vec{z}$. $\hat{\vec{x}} = \frac{\vec{x}\cdot\vec{u_1}}{\vec{u_1}\cdot\vec{u_1}}\vec{u_1} = \frac{14}{7}\vec{u_1} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}$. Then $\vec{z} = \vec{x} - \hat{\vec{x}} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$.

Answer of exercise 19

$\hat{\vec{y}} = \frac{\vec{y}\cdot\vec{u_1}}{\vec{u_1}\cdot\vec{u_1}}\vec{u_1} + \frac{\vec{y}\cdot\vec{u_2}}{\vec{u_2}\cdot\vec{u_2}}\vec{u_2} = \frac{30}{10}\vec{u_1} + \frac{26}{26}\vec{u_2} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$. The distance between $\vec{y}$ and W is $\|\hat{\vec{y}} - \vec{y}\| = \| \begin{bmatrix} -4 \\ -4 \\ -4 \\ -4 \end{bmatrix} \| = \sqrt{64} = 8$.

Answer of exercise 20

We apply the Gram-Schmidt process. $\vec{v_1} = \vec{x_1}$. $\vec{v_2} = \vec{x_2} - \frac{\vec{x_2}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} = \vec{x_2} - \frac{-45}{15}\vec{v_1} = \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$. Thus an orthogonal basis for W is $\{\vec{v_1}, \vec{v_2}\}$.

Answer of exercise 21

$\vec{v_1} = \vec{a_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v_2} = \vec{a_2} - \frac{\vec{a_2}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v_3} = \vec{a_3} - \frac{\vec{a_3}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} - \frac{\vec{a_3}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2} = \vec{a_3} - \frac{14}{4}\vec{v_1} - \frac{12}{8}\vec{v_2} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}$. Thus an orthogonal basis for W is $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

Answer of exercise 22

Gram-Schmidt gives us an orthogonal basis for the columnspace of A which is $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \}$ normalizing and placing these vectors in a matrix gives us $Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}$. Multiplying A with Q^T gives us $R = Q^T A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$.

Answer of exercise 23

We have to solve $A^T A = A^T \vec{b}$. $\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$, row reducing gives us $\hat{\vec{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For the least squares error we need to calculate $\|A\hat{\vec{x}}-\vec{b}\| = \| \begin{bmatrix}4\\0\\2\end{bmatrix} - \begin{bmatrix}5\\1\\0\end{bmatrix} \| = \| \begin{bmatrix}-1\\-1\\2\end{bmatrix} \| = \sqrt{6}$.

Answer of exercise 24

We have to solve $A^TA = A^T\vec{b}$. $\begin{bmatrix}5 & -1\\-1 & 5\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$, row reducing gives us $\hat{\vec{x}} = \begin{bmatrix}1/2\\1/2\end{bmatrix}$.

Answer of exercise 25

Make design matrix $X = \begin{bmatrix}1 & 2\\1 & 3\\1 & 5\\1 & 6\end{bmatrix}$ and observation vector $\vec{Y} = \begin{bmatrix}3\\2\\1\\0\end{bmatrix}$. Now we need to calculate the least squares solution. So calculate $X^TX = \begin{bmatrix}4 & 16\\16 & 74\end{bmatrix}$ and $X^T\vec{Y} = \begin{bmatrix}6\\17\end{bmatrix}$. Row reduction gives us $\vec{\beta} = \begin{bmatrix}4.3\\-0.7\end{bmatrix}$. Thus the line is $y = 4.3 - 0.7x$.

Answer of exercise 26

$\langle p,q\rangle = p(-2)q(-2)+p(-1)q(-1)+p(0)q(0) = (-14)(0)+(-4)(2)+(0)(4) = -8$. $\|p\| = \sqrt{\langle p,p\rangle} = \sqrt{(-14)^2+(-4)^2+0^2} = \sqrt{212}$. $\|q\| = \sqrt{\langle q,q\rangle} = \sqrt{0^2+2^2+4^2} = \sqrt{20}$. $proj_p q = \frac{\langle p,q\rangle}{\langle p,p\rangle}p = \frac{-8}{212}p = \frac{-8}{212}(t-3t^2)$.

Answer of exercise 27

$\langle f,g\rangle = \frac{1}{1+1}\int_{-1}^{1}(t+t^2)(1-3t)dt = \frac{1}{2}\int_{-1}^{1} t-2t^2-3t^3dt = \frac{1}{2}(\frac{1}{2}t^2-\frac{2}{3}t^3-\frac{3}{4}t^4|_{-1}^{1}) = -\frac{2}{3}$.

Answer of exercise 28

We need to find the eigenvectors corresponding to the given eigenvalues. Straightforward calculations gives us $\vec{v_1} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$, $\vec{v_2} = \begin{bmatrix}1\\-2\\1\end{bmatrix}$ and $\vec{v_3} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Normalizing and placing them in a matrix gives us $P = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}\\0 & -2/\sqrt{6} & 1/\sqrt{3}\\1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}$ placing the corresponding eigenvectors in a matrix gives us $D = \begin{bmatrix}-4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 7\end{bmatrix}$. $P^{-1} = P^T$.

Answer of exercise 29

We need to find the eigenvectors. Calculations will give us $\vec{v_1} = \begin{bmatrix}1\\1/2\\-1\end{bmatrix}$ corresponding to -2.

$\vec{v_2} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\vec{v_3} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ corresponding to 7. The last two eigenvectors are not orthogonal, so apply Gram-Schmidt to get $\vec{v_3}' = \begin{bmatrix} 4/5 \\ 2/5 \\ 1 \end{bmatrix}$. Normalizing these vectors and placing them in a matrix gives us $P = \begin{bmatrix} 1/\sqrt{5} & 4/3 \times \sqrt{5} & 2/3 \\ -2/\sqrt{5} & 2/3 \times \sqrt{5} & 1/3 \\ 0 & 5/3 \times \sqrt{5} & -2/3 \end{bmatrix}$. $D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

Answer of exercise 30

Because A is symmetric we know that the quadratic form is $3x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$. In this case is that equal to 66.

Answer of exercise 31

This can be written as the following symmetric matrix. $A = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 0 & 1/2 \\ 0 & 1/2 & 2 \end{bmatrix}$.

Answer of exercise 32

The matrix of the quadratic form is $A = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$. Calculations will show that the eigen values are 7 and 2 therefore the quadratic form is positive definite. The eigenvectors are $\vec{v_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ respectively. Scaling these will give $\vec{u_1} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $\vec{u_2} = \vec{v_2} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. With these we can make a matrix P that changes A into D. $P = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ and $D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$. If $\vec{x} = P\vec{y}$ then the new quadratic form is $7y_1^2 + 2y_2^2$.

Answer of exercise 33

The matrix of this quadratic form is $A = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. The eigenvalues of this matrix are 0, 4 and 7. The maximum of $Q(\vec{x})$ is equal to the greatest eigenvalues, thus 7 this is attained for the unit eigenvector corresponding to 7, which is $\vec{u} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The maximum under the second contraints is the second largest eigenvalue 4.

Answer of exercise 34

$A^T A = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$ whose eigenvalues are $\lambda_1 = 90$ and $\lambda_2 = 0$. The eigenvectors are $\vec{v_1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Normalizing gives us $\vec{v_1}' = \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$ and $\vec{v_2}' = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$. Now we can construct $V = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$. The singular values are $\sigma_1 = \sqrt{90} = 3\sqrt{10}$ and $\sigma_2 = 0$. Now we can construct $\Sigma = \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\vec{u_1} = \frac{1}{\sigma_1} A\vec{v_1}' = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$. Because $\sigma_2 = 0$ we have to find $\vec{u_2}$ and $\vec{u_3}$ by extending $\{\vec{u_1}\}$ to a orthonormal basis for $\mathbb{R}^3$. Thus $\vec{u_1}^T \vec{x} = 0$ which gives us $\vec{w_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{w_3} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ applying Gram-Schmidt and normalizing gives us $\vec{u_2} = \begin{bmatrix} 2/3 \\ -1/2 \\ 2/3 \end{bmatrix}$ and $\vec{u_3} = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$. Thus $A = U\Sigma V^T = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$.