Vrije Universiteit Amsterdam **March 28, 2024**
X_400042: Linear Algebra – Midterm Exam **18:45-21:00**
Lecturer: İlke Çanakçı

Instructions:

- Show ALL of your calculations and display answers clearly.
- Please write neatly and make sure your handwriting is legible. Cross out any work that you do not wish to be considered for grading.
- Please *justify* your answers! Even a correct answer without full explanation scores badly (e.g. indicate the theorems you use, add necessary calculations, etc.).
- The use of books, lecture notes, calculators, electronic devices, etc. is *not* allowed.
- **Your grade = points/10 +1.**

Question 1 [*20 pts*]

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 8 \\ 1 & 2 & -1 & 1 \end{bmatrix} = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}\ \mathbf{a_4}]$ and the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

(a) [*3 pts*] Find the reduced echelon form of A.

(b) [*2 pts*] Find a basis for $Col(A)$, the column space of A.

(c) [*2 pts*] Express $\mathbf{a_4}$ as a linear combination of $\mathbf{a_1}$ and $\mathbf{a_3}$.

(d) [*2 pts*] What is the rank of A?

(e) [*2 pts*] Find the dimension of the null space of A.

(f) [*5 pts*] Find a basis for the null space of A.

(g) [*4 pts*] Find a general solution to the linear system $A\mathbf{x} = \mathbf{b}$, if it exists.

Question 2 [*10 pts*] Consider the matrix $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}\ \mathbf{a_4}]$.

(a) [*6pts*] Determine an LU-factorisation of A.

(b) [*4pts*] Does the columns of A span $\mathbb{R}^3$?

Question 3 [*15 pts*] Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}]$.

(a) [*4pts*] Justify why A is an invertible matrix.

(b) [*7pts*] Find the inverse of A.

(c) [*4pts*] Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and find the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B} = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$.

Exam continues on page 2!

Question 4 [*15 pts*]

Consider the transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ given by $T(a + b\,t + c\,t^2) = \begin{bmatrix} a+b \\ b+c \end{bmatrix}$, where $\mathbb{P}_2$ is the vector space of polynomials of degree at most 2.

(a) [*5 pts*] Show that T is linear.

(b) [*5 pts*] Describe $ker(T)$ as a span of linearly independent vectors in $\mathbb{P}_2$.

(c) [*2 pts*] Is T one-to-one?

(d) [*3 pts*] Is T onto?

Question 5 [*15 pts*]

Consider the following polynomials in $\mathbb{P}_2$, the vector space of polynomials of degree at most 2.

$$\begin{aligned} p_1(t) &= 1 - t \\ p_2(t) &= 1 + t^2 \\ p_3(t) &= 2 + t - t^2 \\ p_4(t) &= 3 - 4\,t + 3\,t^2 \end{aligned}$$

(a) [*8 pts*] Prove that the set $\mathcal{B} = \{p_1, p_2, p_3\}$ forms a basis for $\mathbb{P}_2$.

(b) [*7 pts*] Find the coordinates of p_4 in the basis $\mathcal{B}$ from part (a).

Question 6 [*15 pts*]

Decide whether the following statements are *true* or *false*. Explain your answer, either by giving a proof if the statement is true, or by giving a proof or providing a counterexample if the statement is false.

(a) [*5 pts*] Let A be an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear transformation whose standard matrix is A. If the number of pivot positions in A equals the number of columns, then T is surjective.

(b) [*5 pts*] If A^2 is an invertible matrix, then A is invertible.

(c) [*5 pts*] If $T : V \to W$ is a map between vector spaces V and W and $T(0) = 0$, then T is linear.

End of exam