LA - Exam 1 ' 2024

Solutions

1 (a) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 8 \\ 1 & 2 & -1 & 1 \end{bmatrix} \underset{\boxed{r_3 - r_1}}{\overset{\boxed{r_2 - 2r_1}}{\sim}} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \overset{r_3 + r_2}{\sim} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) Pivot colums of A form a basis for Col A so $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis.

(c) From the reduced echelon form of A we find that $\underline{a}_4 = 3\underline{a}_1 + 2\underline{a}_3$.

(d) rank A = dim (Col A) = 2 by (b).

(e) rank A + dim (Nul A) = $n = 4$ so dim (Nul A) = $4 - 2 = 2$.

(f) Nul $A = \{ \underline{x} \in \mathbb{R}^4 \mid A\underline{x} = \underline{0} \}$. By part (a), solutions to $A\underline{x} = \underline{0}$ satisfy

$$\begin{cases} x_1 + 2x_2 + 3x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = -2x_2 - 3x_4 \\ x_2 = s \text{ free} \\ x_3 = -2x_4 \\ x_4 = t \text{ free} \end{cases}$$

$$\text{Nul}(A) = \left\{ \begin{bmatrix} -2s - 3t \\ s \\ -2t \\ t \end{bmatrix}, \ s, t \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \quad \text{so} \quad \mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

is a basis for Nul A.

(g) $[A \mid \underline{b}] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & 1 & 8 & 2 \\ 1 & 2 & -1 & 1 & 2 \end{array}\right] \underset{\boxed{r_3 - r_1}}{\overset{\boxed{r_2 - 2r_1}}{\sim}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 1 \end{array}\right] \overset{r_3 + r_2}{\sim} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

The last row implies $0 = 1$, contradiction. Thus, the system $A\underline{x} = \underline{b}$ does not have any solutions.

② ⓐ

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \underset{r_3 - r_1}{\overset{r_2 - 2r_1}{\sim}} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \overset{r_3 - r_2}{\sim} \underbrace{\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{= U}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } A = LU.$$

ⓑ Note that U is an echelon form of A and so A has 3 pivot positions. Then the columns of A span $\mathbb{R}^3$.

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③ ⓐ A is invertible if and only if $\det A \neq 0$. Since we have

$$\det A = \begin{vmatrix} +1 & -0 & 1 \\ -1 & +0 & 2 \\ 2 & -1 & 0 \end{vmatrix} \overset{\text{through column 2}}{=} -1 \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -\left(2(1) - 1(-1)\right) = -3 \neq 0,$$

Thus, A is invertible.

ⓑ

$$[A \,\vdots\, I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{r_3 - 2r_1}{\overset{r_2 + r_1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{r_2 \longleftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \\ 0 & 0 & 3 & 1 & 1 & 0 \end{array}\right] \overset{r_3/3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1/3 & 1/3 & 0 \end{array}\right]$$

$$\underset{r_2 + 2r_3}{\overset{r_1 - r_3}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & 0 \\ 0 & 1 & 0 & -4/3 & 2/3 & 1 \\ 0 & 0 & 1 & 1/3 & 1/3 & 0 \end{array}\right] = [I_3 \,\vdots\, A^{-1}]$$

where the right block is A^{-1}.

Alternatively, use Cramer's rule to find A^{-1}.

(c) We have the change of coordinate matrix $P_B = A$ and

$\underline{x} = P_B [\underline{x}]_B = A [\underline{x}]_B$. Then

$$[\underline{x}]_B = A^{-1}\underline{x} = \begin{bmatrix} 2/3 & -1/3 & 0 \\ -4/3 & 2/3 & 1 \\ 1/3 & 1/3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 - 2/3 \\ -4/3 + 4/3 + 1 \\ 1/3 + 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Alternatively, solve for $A\underline{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

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(4) (a) $T(p+q) = T(p) + T(q)$

For $p(t) = a+bt+ct^2$, $q(t) = d+et+ft^2 \in \mathbb{P}_2$, we have

$T(p) = \begin{bmatrix} a+b \\ b+c \end{bmatrix}$ and $T(q) = \begin{bmatrix} d+e \\ e+f \end{bmatrix}$ so $T(p) + T(q) = \begin{bmatrix} a+b+d+e \\ b+c+e+f \end{bmatrix}$.

Since $p(t)+q(t) = (a+d) + (b+e)t + (c+f)t^2$, we have

$$T(p+q) = \begin{bmatrix} a+d+b+e \\ b+e+c+f \end{bmatrix} = T(p) + T(q).$$

$T(\alpha p) = \alpha T(p)$, $\alpha \in \mathbb{R}$, $\alpha p(t) = (\alpha a) + (\alpha b)t + (\alpha c)t^2$

so $T(\alpha p) = \begin{bmatrix} \alpha a + \alpha b \\ \alpha b + \alpha c \end{bmatrix} = \alpha \begin{bmatrix} a+b \\ b+c \end{bmatrix} = \alpha T(a+bt+ct^2)$.

Thus T is linear.

(b) $\ker T = \left\{ a+bt+ct^2 \in \mathbb{P}_2 : T(a+bt+ct^2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$$T(a+bt+ct^2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} a+b \\ b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} a+b = 0 \\ b+c = 0 \end{cases}$$

So $a=-b$ and $b=-c$. Therefore $a=-b=c$. Thus

$$\ker T = \{ a - at + at^2 \in \mathbb{P}_2 : a \in \mathbb{R} \} = \text{Span}\{1 - t + t^2\}$$

(c) T is not 1-1 since $\ker T \neq \{0\}$.

(d) T is onto since for any $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, $T(a + bt^2) = \begin{bmatrix} a \\ b \end{bmatrix}$.

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(5) (a) Consider the coordinate mapping $T: \mathbb{P}_2 \longrightarrow \mathbb{R}^3$ where $p \longmapsto [p]$

$[p]$ denotes the coordinates of p in the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. Then

$$A = \big[\,[p_1]\ [p_2]\ [p_3]\,\big] = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \overset{r_2+r_1}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & -1 \end{bmatrix} \overset{r_3-r_2}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix}$$

3 pivot positions $\Longrightarrow$ columns form a basis for $\mathbb{R}^3$

$\Longrightarrow p_1, p_2, p_3$ form a basis for $\mathbb{P}_2$.

(b) The coordinates of p_4 in $\mathcal{B} = \{p_1, p_2, p_3\}$ is given by the weights of the equation $c_1[p_1] + c_2[p_2] + c_3[p_3] = [p_4]$. Then

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ -1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 3 \end{array}\right] \overset{r_2+r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{array}\right] \overset{r_3-r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -4 & 4 \end{array}\right]$$

$$\Longrightarrow \begin{cases} c_1 + c_2 + 2c_3 = 3 \\ c_2 + 3c_3 = -1 \\ -4c_3 = 4 \end{cases} \longrightarrow \begin{cases} c_3 = -1 \\ c_2 - 3 = -1 \Rightarrow c_2 = 2 \\ c_1 + 2 - 2 = 3 \Rightarrow c_1 = 3 \end{cases}$$

$$\Longrightarrow \boxed{[\underline{p}_4]_B = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}}$$

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6 (a) FALSE: $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T(\underline{x}) = A\underline{x}$. Then T is surjective if and only if A has a pivot in every row. However, A has a pivot in every column. Thus, if # columns is less than # rows, T is not surjective.

(b) TRUE: If A^2 invertible, $\det A^2 = (\det A)^2 \neq 0$. Then $\det A \neq 0$. That is, A is invertible.

(c) FALSE: Consider $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $\begin{bmatrix} a \\ b \end{bmatrix} \longmapsto \begin{bmatrix} a^2 \\ 0 \end{bmatrix}$. Then $T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. But T is not linear since $T\begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix} = \begin{bmatrix} (\alpha a)^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha^2 a^2 \\ 0 \end{bmatrix}$ but $\alpha T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a^2 \\ 0 \end{bmatrix}$ so $T\left(\alpha \begin{bmatrix} a \\ b \end{bmatrix}\right) \neq \alpha T\begin{bmatrix} a \\ b \end{bmatrix}$.