Vrije Universiteit Amsterdam **July 5, 2022**
Linear Algebra – Resit Exam **8:30-11:15**
Lecturer: Jose Mujica

Instructions:
Please write your answers clearly.
Motivate your answers (e.g. indicate the theorems you use, add necessary calculations, etc.).
No books/calculators/computers allowed.
Your grade = points/6 +1.

Question 1 (3p,2p,3p)

Let $A = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}\ \mathbf{a_4}\ \mathbf{a_5}]$ be a 4×5 matrix with columns $\mathbf{a_1}, \ldots, \mathbf{a_5}$, and $\mathbf{b}$ a 4×1 matrix such that the augmented matrix $[A|\mathbf{b}]$ row reduces to

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Which columns of A form a basis for $Col(A)$, the column space of A? what is the dimension of $Col(A)$? Express the remainder columns of A as linear combinations of the vectors of that basis.

(b) What is the rank of A? Use your answer to deduce the dimension of $Nul(A)$, the null space of A.

(c) Find the general solution to the linear system $A\mathbf{x} = \mathbf{b}$.

Question 2 (3p,3p)

Consider the matrix $B = \begin{bmatrix} 2 & 6 & -2 & -4 & 4 \\ 1 & -3 & 3 & 0 & 2 \\ -3 & -3 & 1 & 1 & 0 \end{bmatrix}$

(a) Give the LU-factorization of B.

(b) Find a basis for the null space of B.

Question 3 (2p,3p,3p)

Consider the matrix $A = \begin{bmatrix} 2 & k \\ -2 & -2 \end{bmatrix}$, with $k \in \mathbb{R}$.

(a) For which values of $k \in \mathbb{R}$ does A have real eigenvalues?

(b) For which values of k found in (a) is A diagonalizable? Explain your answer.

(c) Let $k = 0$. Find 2×2 matrices P invertible and D diagonal such that $A = PDP^{-1}$.

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Question 4 (3p,3p,2p)

Consider the matrix $C = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 2 & 5 \\ 0 & 2 & 4 \\ -4 & -1 & -3 \end{bmatrix}$ and the vector $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

(a) Find an orthogonal basis for the column space of C.

(b) Compute the orthogonal projection of $\mathbf{y}$ onto the column space of C.

(c) Find a QR factorization of C.

Question 5 (4p,6p)

Consider the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \\ -2 & 2 \end{bmatrix}$

(a) Orthogonally diagonalize $A^T A$.

(b) Give a singular value decomposition of A.

Question 6 (3p,2p,3p)

Let $H = \{\mathbf{p} \in \mathbb{P}_2 : \mathbf{p}(0) = 0\}$ be the set of all polynomials of degree at most two that pass through the origin.

(a) Prove that H is a subspace of $\mathbb{P}_2$.

Consider the transformation $T : \mathbb{R}^2 \to \mathbb{P}_2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 t + 2x_2 t^2$.

(b) Prove that T is a linear transformation.

(c) Find the matrix of T relative to the standard basis $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ of $\mathbb{R}^2$ and the basis $\mathcal{C} = \{2t, t - t^2\}$ of H.

Question 7 (2p,2p,2p)

Decide whether the following statements are *true* or *false*. Explain your answer by giving a proof if the statement is true, or by giving a proof or providing a counterexample if the statement is false.

(a) Let A be an $m \times n$ matrix. if $m < n$, then the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for all $\mathbf{b} \in \mathbb{R}^m$.

(b) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal set of vectors in $\mathbb{R}^n$, then $||\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3|| = \sqrt{3}$.

(c) If A and B are $n \times n$ matrices such that AB and A are both invertible, then B is also invertible.

-END OF EXAM-