Solution resit

1

(a) From the reduced form the l.i. columns are a_1, a_3, a_4. Therefore

$B = \{a_1, a_3, a_4\}$ is a basis for $Col(A)$

From here, $\dim(Col(A)) = 3$.

We can write the remainder columns as

$a_2 = 2a_1$

$a_5 = a_1 + a_3 + 2a_4$

(b) From (a), Rank(A) = 3.

From the dimension thm,

$$\text{Rank}(A) + \dim(\text{Nul}(A)) = 5$$

$$\therefore \dim(\text{Nul}(A)) = 2$$

(c) From the echelon form we have,

for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$,

$$x_1 = 1 - 2x_2 - x_5$$

$$x_3 = 1 - x_5 \qquad , x_2, x_5 \in \mathbb{R}.$$

$$x_4 = -1 - 2x_5$$

2]

(a) $$B = \begin{bmatrix} 2 & 6 & -2 & -4 & 4 \\ 1 & -3 & 3 & 0 & 2 \\ -3 & -3 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 6 & -2 & -4 & 4 \\ 0 & -6 & 4 & 2 & 0 \\ 0 & 6 & -2 & 5 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 6 & -2 & -4 & 4 \\ 0 & -6 & 4 & 2 & 0 \\ 0 & 0 & 2 & -3 & 6 \end{bmatrix} = U$$

From here

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1 & 0 \\ -3/2 & 1 & 1 \end{bmatrix}$$

(b)

$$B \sim U \sim \begin{bmatrix} 1 & 0 & 0 & 1/2 & -1 \\ 0 & 1 & 0 & -4/3 & 2 \\ 0 & 0 & 1 & -3/2 & 3 \end{bmatrix}$$

if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \text{Nul}(B)$, then

$$x = x_4 \begin{bmatrix} -1/2 \\ 4/3 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_4, x_5 \in \mathbb{R}.$$

A basis for Nul(B) is then

$$\mathcal{B} = \left\{ \begin{bmatrix} -1/2 \\ 4/3 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3]

(a) $p(\lambda) = \lambda^2 - 4 + 2k = 0$

$$\iff \lambda^2 = 4 - 2k$$

From here, $\lambda \in \mathbb{R} \iff k \leq 2$

(b) If $k < 2$, $\lambda = \pm\sqrt{4-2k}$

and we have two different real eigenvalues, which means l.i. eigenvectors and A diagonalizable.

If $k=2$, $A=\begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$

has $\lambda=0$ with multiplicity 2.

let's find eigenvectors:

$$A-0\cdot I_2=\begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and $v=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Therefore

$\dim(E_{\lambda=0})=1$ and A is not diagonalizable. Therefore, A is diagonalizable for $k<2$.

c) For $k=0$, $A = \begin{bmatrix} 2 & 0 \\ -2 & -2 \end{bmatrix}$

has eigenvalues $\lambda_1 = 2$, $\lambda_2 = -2$.

For $\lambda_1 = 2$ an eigenvector is $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For $\lambda_2 = -2$ an eigenvector is $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

From here,

$P = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$; $D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

4)

$$C = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 2 & 5 \\ 0 & 2 & 4 \\ -4 & -1 & -3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

a) $v_1 = x_1$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} x_1 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix} - \frac{18}{36} \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= \begin{bmatrix} 2 \\ 5 \\ 4 \\ -3 \end{bmatrix} - \frac{36}{36} \begin{bmatrix} 2 \\ 4 \\ 0 \\ 4 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

an orthogonal basis is then $B = \{v_1, v_2, v_3\}$

b) $$\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{y \cdot v_3}{v_3 \cdot v_3} v_3$$

$$= \frac{-4}{36} \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix} + \frac{2}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

c) $$Q = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 0 & 1/3 \\ 0 & 2/3 & 2/3 \\ -2/3 & 1/3 & 0 \end{bmatrix}$$

$$C = QR \Rightarrow R = Q^T C = \begin{bmatrix} 6 & 3 & 6 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

5] $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ -2 & 2 \end{bmatrix}$

a) $A^T A = \begin{bmatrix} 8 & 0 \\ 0 & 12 \end{bmatrix}$

The eigenvalues are

$\lambda_1 = 8, \quad \lambda_2 = 12$

with corresponding eigenvectors

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(already orthonormal!)

We can write $A^T A = PDP^T$,

with $D = \begin{bmatrix} 8 & 0 \\ 0 & 12 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) The singular values are

$\sigma_1 = \sqrt{12}$, $\sigma_2 = \sqrt{8}$. Then

$\Sigma = \begin{bmatrix} \sqrt{12} & 0 \\ 0 & \sqrt{8} \\ 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$Av_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \implies \frac{Av_1}{\sigma_1} = \frac{2}{\sqrt{12}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \implies \frac{Av_2}{\sigma_2} = \frac{2}{\sqrt{8}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Need to find $w \in \mathbb{R}^3$ such that

$$w \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = w \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0.$$

From here, $w = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and we

can choose $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ to complete $\mathbb{R}^3$.

From here

$$U = \begin{bmatrix} 2/\sqrt{12} & 2/\sqrt{8} & 1/\sqrt{6} \\ 2/\sqrt{12} & 0 & -2/\sqrt{6} \\ 2/\sqrt{12} & -2/\sqrt{8} & 1/\sqrt{6} \end{bmatrix}$$

and we have

$$A = U \Sigma V^T.$$

6) $H = \{ p \in \mathbb{P}_2 : p(0) = 0 \}$

a) (i) Clearly $p \equiv 0 \in H$.

(ii) let $p, q \in H$. then

$$(p+q)(0) = p(0) + q(0)$$
$$= 0 + 0 = 0$$

$\therefore p+q \in H$

(iii) let $p \in H$, $c \in \mathbb{R}$.

$$cp(0) = c \cdot p(0) = c \cdot 0 = 0$$

$\therefore cp \in H$.

From (i), (ii) and (iii), H is subspace of $\mathbb{P}_2$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{P}_2 \;;\; T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 t + 2x_2 t^2$$

(i) let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$.

$$T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = (x_1 + y_1)t + 2(x_2 + y_2)t^2$$

$$= x_1 t + 2x_2 t^2 + y_1 t + 2y_2 t^2$$

$$= T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$

(ii) Let $c \in \mathbb{R}$.

$$T\left(c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) = cx_1 t + 2cx_2 t^2$$

$$= c(x_1 t + 2x_2 t^2)$$

From (i) and (ii) T is linear.

c) Let $C = \{2t, t - t^2\} = \{p_1, p_2\}$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = t = \frac{1}{2} p_1 + 0 \cdot p_2$$

so $$\left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right]_C = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 2t^2 = 1 \cdot p_1 - 2 \cdot p_2$$

and $$\left[T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right]_C = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Therefore

$$T_A = \begin{bmatrix} [T(e_1)]_c & [T(e_2)]_c \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1 \\ 0 & -2 \end{bmatrix}.$$

7)

a) False.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

is inconsistent

b) True.

$$\|v_1 + v_2 + v_3\|^2 = (v_1+v_2+v_3)\cdot(v_1+v_2+v_3)$$

$$= v_1\cdot v_1 + v_2\cdot v_2 + v_3\cdot v_3 + 2\underbrace{v_1\cdot v_2}_{=0} + 2\underbrace{v_1\cdot v_3}_{=0} + 2\underbrace{v_2\cdot v_3}_{=0}$$

$$= 3$$

c) True.

A invertible $\Rightarrow$ A^{-1} invertible

AB invertible $\Rightarrow$ $A^{-1} \cdot AB$ invertible

$\Rightarrow$ B invertible.