Vrije Universiteit Amsterdam **May 30, 2022**
Linear Algebra – Final Exam **8:30-10:30**
Lecturer: Jose Mujica

Instructions:
Please write your answers clearly.
Motivate your answers (e.g. indicate the theorems you use, add necessary calculations, etc.).
No books/calculators/computers allowed.
Your grade = points/10 +1.

Question 1 (5p,5p,5p,5p)

Consider the matrix $A = \begin{bmatrix} 1 & k \\ -1 & -1 \end{bmatrix}$, with $k \in \mathbb{R}$.

(a) For which values of $k \in \mathbb{R}$ does A have real eigenvalues?

(b) For which values of k found in (a) is A diagonalizable? Explain your answer.

(c) Let $k = 0$. Find 2×2 matrices P invertible and D diagonal such that $A = PDP^{-1}$.

(d) Let $k = 2$. Find a 2×2 invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$.

Question 2 (5p,5p, + 5p bonus)

Consider the discrete dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, with $A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$.

(a) Show that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors of A. What are their corresponding eigenvalues?

(b) Let $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ be the initial state of the system. Find an explicit formula for $\mathbf{x}_k$ in terms of k, the eigenvalues and the eigenvectors of A and describe the behavior of $\mathbf{x}_k$ as $k \to \infty$.

(c) **Optional (Bonus!)**. Note that the coordinates of the initial state of the system $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ satisfy $u_1 < u_2$. Given your answer in (b), can the coordinates of $\mathbf{x}_k$ satisfy $\underline{u_1 > u_2}$ for some k?

Question 3 (5p,5p,10p)

Consider the matrix $C = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$ and the vector $\mathbf{y} = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$

(a) Find an orthogonal basis for the column space of C.

(b) Compute the orthogonal projection of $\mathbf{y}$ onto the column space of C.

(c) Determine the least-squares solution of the inconsistent system $C\mathbf{x} = \mathbf{y}$.

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Question 4 (5p,10p)

Consider the matrix $A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \\ 2 & -2 \end{bmatrix}$

(a) Orthogonally diagonalize $A^T A$.

(b) Give a singular value decomposition of A.

Question 5 (5p,5p,5p)

Decide whether the following statements are *true* or *false*. Explain your answer by giving a proof if the statement is true, or by giving a proof or providing a counterexample if the statement is false.

(a) If an $m \times n$ matrix U has orthonormal columns, then $U^T U$ is diagonalizable.

(b) If A is an $n \times n$ matrix which contains a row or column of zeros, then 0 is an eigenvalue of A.

(c) If A is a symmetric $n \times n$ matrix, then $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Note: Here, '$\cdot$' represents the usual *dot* product in $\mathbb{R}^n$.

Question 6 (5p,5p)

Consider the inner product space $\mathbb{P}_2$ of all polynomials of degree less than or equal to 2, with inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(-1)\mathbf{q}(-1) + \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1),$$

and the vectors $\mathbf{p}(t) = 1 + t^2$, $\mathbf{q}(t) = 1 + t$ in $\mathbb{P}_2$.

(a) Find an orthogonal basis for $W = span\{\mathbf{p}, \mathbf{q}\}$.

(b) Calculate the orthogonal projection of $r(t) = -1 + 2t + t^2$ onto W.

-END OF EXAM-