Solution exam 2 LA

Q1) $A = \begin{bmatrix} 1 & k \\ -1 & -1 \end{bmatrix}$

(a) The characteristic equation of A is

$$\lambda^2 - 1 + k = 0$$

$$\iff \quad \lambda^2 = 1 - k$$

From here, A has real eigenvalues if and only if $1-k \geq 0$, that is

$\boxed{k \leq 1}$.

(Rmk: for $k > 1$ A has complex eigenvalues).

(b) If $k<1$, the eigenvalues are

$$\lambda = \pm\sqrt{1-k}.$$ Therefore A

is diagonalizable for $k<1$

(A has 2 different eigenvalues).

For $k=1$, $\lambda=0$ is an eigenvalue

with algebraic multiplicity 2.

We have, for $k=1$

$$A-0\cdot I_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and the corresponding eigenspace is

$$E_\lambda = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$ Since this

subspace is one-dimensional, A is not

diagonalizable. Therefore A is diagonalizable

if $k<1$

(c) For $k=0$, $A = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$

the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$,

with corresponding eigenvectors

$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively.

From here,

$$P = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are matrices that diagonalize A.

(For $k=0$).

d) For $k=2$, $A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

has eigenvalues $\lambda = \pm i$.

For $\lambda = a - bi = -i$, we have

$$A - \lambda I_2 = \begin{bmatrix} 1+i & 2 \\ -1 & -1+i \end{bmatrix} \sim \begin{bmatrix} 1+i & 2 \\ 0 & 0 \end{bmatrix}$$

From where $v = \begin{bmatrix} 2 \\ -1-i \end{bmatrix}$ is an eigenvector (complex).

then

$$P = [\,\mathrm{Re}(v) \quad \mathrm{Im}(v)\,] = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

are such that $A = PCP^{-1}$.

Q2 $A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$

(a) $Av_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$Av_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.3\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So v_1 and v_2 are eigenvectors of A,

with corresponding eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0.3$.

(b) We have

$$x_0 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = v_1 + v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and

$$x_k = A^k x_0 = A^k \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

$$= \lambda_1^k \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \lambda_2^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} + (0.3)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now, when $k \to \infty$ $x_k \longrightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

($\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an attractor).

(c) (Bonus)

given the initial state $x_0 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$,

the system evolves towards $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

along the line $y = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(That is, it follows the direction $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$)

If one moves from $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

along that line, you never cross the

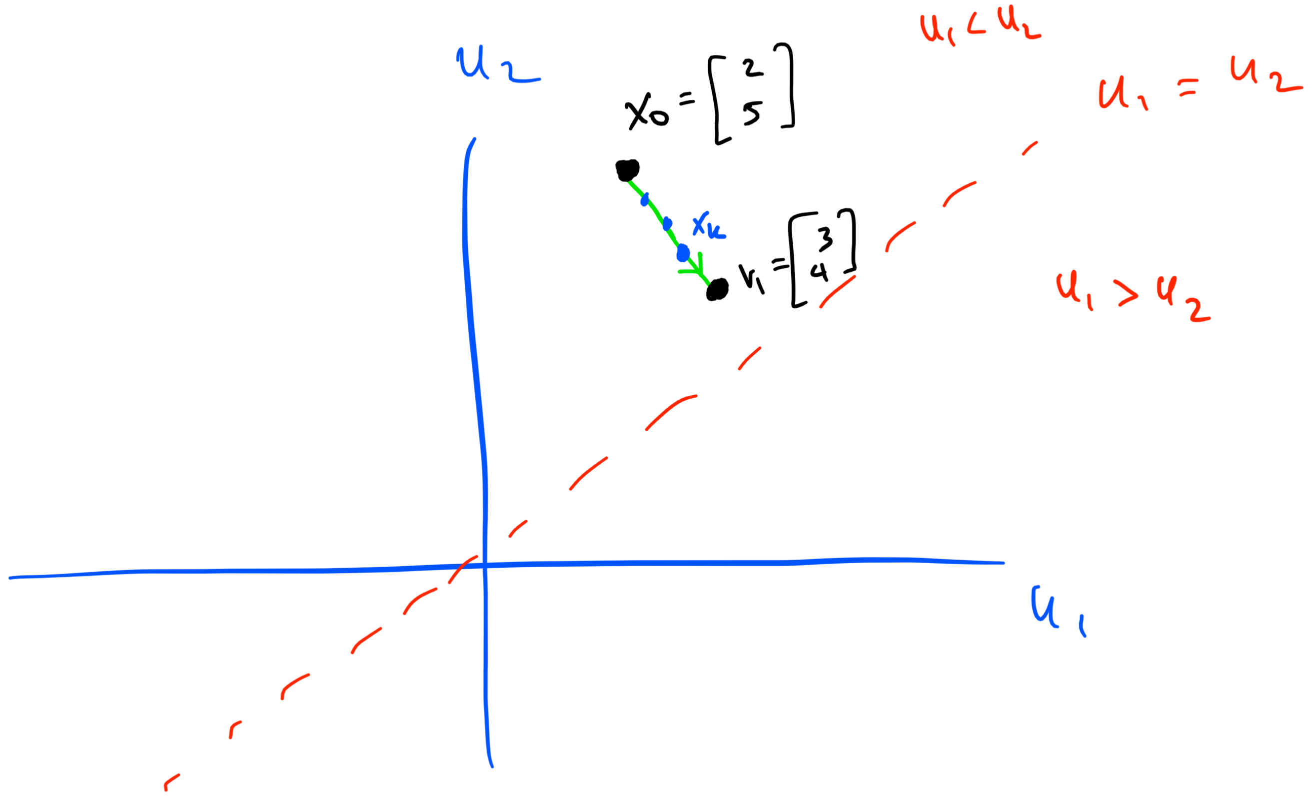
diagonal $\left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : u_1 = u_2 \right\}$, which is

necessary if you want that $u_1 > u_2$

at some point. From here, the condition

$\underline{u_1 > u_2}$ is never satisfied.

Graphic Idea :



Q3

$$C = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = [x_1\ x_2\ x_3], \quad y = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$$

(a)

Note that the columns of C are l.i, but not orthogonal. We use Gram-Schmidt in order to find an orthogonal basis $\{v_1, v_2, v_3\}$:

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 1/4 \\ -3/4 \\ -3/4 \\ 5/4 \end{bmatrix}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 24/11 \\ -17/11 \\ 5/11 \\ -12/11 \end{bmatrix}$$

(b)

$$\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{y \cdot v_3}{v_3 \cdot v_3} v_3$$

$$= 3 v_1 - \frac{48}{11} v_2 + \frac{29}{47} v_3 = \frac{1}{47} \begin{bmatrix} 153 \\ 250 \\ 308 \\ -147 \end{bmatrix}$$

(c) Since the columns of C are l.i. we have a unique least-square solution.

We normalize the equation:

$$C^T C \hat{x} = C^T y$$

$$\begin{bmatrix} 4 & 7 & 10 \\ 7 & 15 & 21 \\ 10 & 21 & 38 \end{bmatrix} \hat{x} = \begin{bmatrix} 12 \\ 9 \\ 20 \end{bmatrix}$$

and obtain

$$\hat{x} = \frac{1}{47}\begin{bmatrix} 492 \\ -242 \\ 29 \end{bmatrix}$$

$\ulcorner$ Alternatively one can find the coordinates of $\hat{y}$ obtained in (b) in terms of the columns of C $\lrcorner$

Q4)

$$A = \begin{bmatrix} -1 & 1 \\ 2 & -2 \\ 2 & -2 \end{bmatrix}$$

a) $A^T A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$ has

eigenvalues $\lambda_1 = 18$

$\lambda_2 = 0.$

For $\lambda_1 = 18$

$$A - 18I = \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and $v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector (normalized)

For $\lambda_2 = 0$

$$A - 0 \cdot I = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

and $v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector (normalized)

From here we can write

$$A^T A = P D P^T, \quad \text{with}$$

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} 18 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) The singular values are

$\sigma_1 = \sqrt{18}$, $\sigma_2 = 0$, then

$$\Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$ Now

$$Av_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 2 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix}$$

and we have

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

we now need to find

$$u_1^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x \cdot u_1 = 0 \right\}$$

we solve $x \cdot u_1 = 0$, that is

$-x+2y+2z=0$ and

obtain $u_1^\perp = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$

We need an orthogonal basis for $u_1^\perp$;

$\{w_2, w_3\}$ where (Gram - Schmidt)

$$w_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$w_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}.$$

Here we can choose $w_3 = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}.$

We normalize these vectors to

obtain u_2 and u_3, columns of U.

$$u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \frac{1}{\sqrt{45}}\begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}.$$

Now we can write

$$A = U \Sigma V^T, \quad \text{with}$$

$$U = \begin{bmatrix} -1/3 & 2/\sqrt{5} & 2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & -4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Q5)

a) True. If U has orthonormal columns $U^T U = I_n$ is already diagonal and, therefore, diagonalizable.

b) True. If A is $n \times n$ and has a row or column of zeros, then $\det(A) = 0$. From the IMT, $\lambda = 0$ is an eigenvalue of A.

c) True.

$$(Ax) \cdot y = (Ax)^T y$$

$$= x^T A^T y$$

$$= x^T A y \quad (A \text{ symmetric})$$

$$= x \cdot (Ay)$$

Q6)

(a) Note that $p(t) = 1+t^2$ and $q(t) = 1+t$ are not orthogonal with this inner product, since

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

$$= 0 + 1 + 4 = 5 \neq 0.$$

An orthogonal basis for $W = \text{span}\{p, q\}$ is given by $\{v_1, v_2\}$, where

$$v_1 = p = 1+t^2$$

$$v_2 = q - \frac{\langle q, p\rangle}{\langle p, p\rangle} p = 1+t - \frac{5}{9}(1+t^2)$$

$$= \frac{4}{9} + t - \frac{5}{9}t^2$$

(b) The orthogonal projection of

$r(t) = -1 + 2t + t^2$ onto W

is given by

$$\hat{r} = \frac{\langle r, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle r, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$= \frac{-1}{9}(1 + t^2) + \frac{32/9}{180/81}\left(\frac{4}{9} + t - \frac{5}{9}t^2\right)$$

$$= -\frac{1}{9}(1 + t^2) + \frac{8}{5}\left(\frac{4}{9} + t - \frac{5}{9}t^2\right)$$

$$= \frac{27}{45} + \frac{8}{5}t - t^2$$