Vrije Universiteit Amsterdam **April 28, 2022**
Linear Algebra – test 4
Lecturer: Jose Mujica

Instructions:
Please write your answers clearly *on a separate paper.* Write down your name and student number.
Motivate your answers (e.g. indicate the theorems you use, add necessary calculations, etc.).
Use scrap paper to do additional computations if necessary. *No* books/calculators/computers allowed.
You have 40 minutes to complete this test.
Your grade = points +1.

Question 1 (1p,1p)

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Provide an invertible matrix P, and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $AP = PC$.

Question 2 (1p,1p,1p,1p)

Consider the vectors $\mathbf{w_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{w_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w_3} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 4 \end{bmatrix}$. Let $W = \text{Span}\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$.

(a) Show that $\mathcal{B} = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is an orthogonal basis for W.

(b) Find the orthogonal projection of $\mathbf{y}$ onto W.

(c) Decompose $\mathbf{y}$ in the form $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$.

(d) Compute the distance from $\mathbf{y}$ to W.

Question 3 (1p,1p,1p)

Decide whether the following statements are *true* or *false*. Explain your answer by giving a proof if the statement is true or giving a proof or providing a counterexample if the statement is false.

(a) If W is a subspace of $\mathbb{R}^n$ and $\mathbf{v}$ is in both W and $W^{\perp}$, then $\mathbf{v}$ must be the zero vector.

(b) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $\mathbb{R}^n$, then $||\mathbf{u} + \mathbf{v}|| = ||\mathbf{u} - \mathbf{v}||$.

(c) If A is a 2×2 matrix with complex eigenvalues and $\mathbf{x} \in \mathbb{R}^2$, then the points $A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, \ldots$ all lie on the same ellipse.