1)

$$A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$$

(a) let's calculate the eigenvalues of A

$$\det(A - \lambda I_2) = 0$$

$$\lambda^2 - 6\lambda + 13 = 0 \implies \lambda_1 = 3+2i$$
$$\lambda_2 = 3-2i$$

(b) We choose $\lambda = 3-2i$ (written in the form $a-bi$). this way we have

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \quad \text{and}$$

$P = [\,\mathrm{Re}(v) \quad \mathrm{Im}(v)\,]$, where v is a complex eigenvalue associated to λ.

Let's find an eigenvector associated to $\lambda = 3-2i$. We solve

$$\left[\begin{array}{cc|c} 1-(3-2i) & 2 & 0 \\ -4 & 5-(3-2i) & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1+i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and we can choose $v = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$. From here,

$$P = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.$$

The matrices C and P satisfy $AP = PC$.

(Note that P is invertible since $\det(P) \neq 0$).

2)

(a) Note that

$$w_1 \cdot w_2 = w_1 \cdot w_3 = w_2 \cdot w_3 = 0$$

and B is an orthogonal set, and therefore linearly independent. From here, B is an orthogonal basis for W.

(b) $$\hat{y} = \frac{y \cdot w_1}{w_1 \cdot w_1} w_1 + \frac{y \cdot w_2}{w_2 \cdot w_2} w_2 + \frac{y \cdot w_3}{w_3 \cdot w_3} w_3$$

$$= \frac{3}{3} w_1 + \frac{12}{3} w_2 - \frac{3}{3} w_3 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(c) We have that $\hat{y} \in W$ and

$$z = y - \hat{y} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \in W^{\perp}.$$ From here

we can write

$$y = \hat{y} + z.$$

(d) $\operatorname{dist}(y, W) = \|z\| = 2\sqrt{3}$

3]

(a) True.

If $v \in W^{\perp}$, then $v \cdot x = 0$ for all $x \in W$. In particular, since $v \in W$ we have

$$v \cdot v = 0 \iff v = 0$$

(b) True.

$$\|u+v\|^2 = (u+v)\cdot(u+v) = u\cdot u + v\cdot v + 2u\cdot v$$

$$\|u-v\|^2 = (u-v)\cdot(u-v) = u\cdot u + v\cdot v - 2u\cdot v$$

if u and v are orthogonal, $u \cdot v = 0$

and $\|u+v\|^2 = \|u-v\|^2$

$$\Longrightarrow \|u+v\| = \|u-v\|.$$

(c) False,

If A has complex-conjugate eigenvalues, then

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$x \longrightarrow T(x) = Ax$$

involves rotations, but also involves expansion or contraction depending on the modulus of the complex eigenvalue.

If $\lambda \in \mathbb{C}$ is the eigenvalue, we have

$|\lambda| < 1 \iff$ contraction + rotation

$|\lambda| > 1 \iff$ expansion + rotation

$|\lambda| = 1 \iff$ rotation only.

For $x \in \mathbb{R}^2$, the points $Ax, A^2x, A^3x, \ldots$ all lie on the same ellipse only when $|\lambda| = 1$.