1

Let's find the eigenvalues of A:

$$\det(A-\lambda I_3)=0 \Rightarrow \det\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}=0$$

$$\Rightarrow -\lambda(\lambda^2-1)+\lambda+1+\lambda+1=0$$

$$\Rightarrow -\lambda(\lambda-1)(\lambda+1)+2(\lambda+1)=0$$

$$\Rightarrow (\lambda+1)(-\lambda(\lambda-1)+2)=0$$

$$\Rightarrow (\lambda+1)^2(\lambda-2)=0$$

[Alternatively, one can use the hint and write the characteristic polynomial as $p(\lambda)=(\lambda-1)\cdot q(\lambda)$, where $q(\lambda)$ is a quadratic polynomial]

the eigenvalues are

$\lambda=-1$ (algebraic multiplicity $=2$)

$\lambda=2$

Let's describe the corresponding eigenspaces:

For $\lambda = -1$, we need to solve the system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From here, the eigenspace associated to $\lambda = -1$ is

$$E_{\lambda=-1} = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Since $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are l.i.,

we have that $\dim\left(E_{\lambda=-1}\right) = 2$.

Since any eigenvector associated to $\lambda = 2$ has to be l.i. with v_1 and v_2, we can obtain a basis of $\mathbb{R}^3$ formed by eigenvectors of A. Therefore, A is diagonalizable.

To obtain a matrices P invertible and D diagonal such that $A = PDP^{-1}$, we need to find an eigenvector of A associated to $\lambda = 2$.

We solve the system

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

then, the eigenspace associated to $\lambda = 2$ is

$$E_{\lambda=2} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

From here we can choose

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

P is clearly invertible, since the chosen eigenvectors are l.i.

2]

the characteristic equation of A is

$$\det(A - \lambda I_2) = 0.$$

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, this means

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow \quad \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\Rightarrow \quad \lambda^2 - \mathrm{tr}(A)\,\lambda + \det(A) = 0$$

$\square$

3)

(a) True.

If λ is an eigenvalue of A with eigenvector v, we have

$$Av = \lambda v \implies A^k v = A^{k-1}(Av)$$
$$= A^{k-1}(\lambda v)$$
$$= \lambda A^{k-1} v$$
$$= \lambda A^{k-2}(Av)$$
$$= \lambda^2 A^{k-2} v$$
$$\vdots$$
$$= \lambda^k v.$$

therefore, λ^k is an eigenvalue of A^k with eigenvector v.

(b) False.

A can be diagonalizable and have $\det(A) = 0$, which means A not invertible.

An example of this is

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix},$$ which has

zero determinant, eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$, with corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$