Vrije Universiteit Amsterdam **March 30, 2022**
Linear Algebra – Midterm Exam **18:45-20:45**
Lecturer: Jose Mujica

Instructions:
Please write your answers clearly.
Motivate your answers (e.g. indicate the theorems you use, add necessary calculations, etc.).
No books/calculators/computers allowed.
Your grade = points/10 +1.

Question 1 (30p, 3 points each item)

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 4 \\ 2 & -1 & 0 & 2 & 1 & 3 \\ 2 & -1 & 1 & 2 & 1 & 3 \\ 1 & 0 & 0 & 1 & 1 & 4 \end{bmatrix} = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}\ \mathbf{a_4}\ \mathbf{a_5}\ \mathbf{a_6}]$ and the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Construct the augmented matrix $[A|\mathbf{b}]$ and use elementary row operations to obtain its reduced echelon form.

(b) Find a basis for $Col(A)$, the column space of A.

(c) Are the columns of A linearly independent?

(d) What is the rank of A?

(e) Use the result obtained in (d) to deduce the dimension of $Nul(A)$, the null space of A.

(f) Express the columns $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}, \mathbf{a_5}$ and $\mathbf{a_6}$ as linear combinations of the vectors of the basis found in (b).

(g) Find the general solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$

(h) How does your answer in (g) relate to the answer to (e)?

(i) Find the general solution to the linear system $A\mathbf{x} = \mathbf{b}$, expressing your answer in parametric vector form.

(j) Give a vector $\mathbf{c} \in \mathbb{R}^4$ such that the equation $A\mathbf{x} = \mathbf{c}$ has no solution.

Question 2 (15p)

Consider the linear system of equations

$$\begin{cases} x + y + 3z &= 1 \\ -2x + 2y + z &= 2 \\ y + z &= 3 \end{cases}$$

Use Cramer's rule to find z.

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Question 3 (15p, 5 points each item)

$$\text{Let } W = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 - x_2 + x_3 = 0;\ x_2 + x_3 - x_4 = 0 \right\}.$$

(a) Prove that W is a subspace of $\mathbb{R}^4$.

(b) Find a basis of W.

(c) What is the dimension of W?

Question 4 (15p, 5 points each item)

Consider the transformation $T : \mathbb{R}^4 \to \mathbb{P}_2$ given by $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (a+b)t^2 + bt + c - d.$

(a) Prove that T is linear.

(b) Describe $ker(T)$ as a span of vectors.

(c) Describe the range of T.

Question 5 (15p, 5 points each item)

Decide whether the following statements are *true* or *false*. Explain your answer by giving a proof if the statement is true or giving a proof or providing a counterexample if the statement is false.

(a) If A and B are $n \times n$ matrices such that AB is invertible, then A and B are both invertible.

(b) Let U, V be vector spaces and $T : V \to U$ a linear transformation. If $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n} \in V$ are such that $T(\mathbf{v_1}), T(\mathbf{v_2}), \ldots, T(\mathbf{v_n}) \in U$ are linearly independent, then $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}$ are also linearly independent.

(c) Let V be a vector space of dimension n. If B and C are bases of V, then the change-of-coordinates matrix from B to C has rank n.

-END OF EXAM-