Solution mid-term exam Linear Algebra 2022

1)

(a) $$[A|b] = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 1 & 4 & 1 \\ 2 & -1 & 0 & 2 & 1 & 3 & 2 \\ 2 & -1 & 1 & 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 1 & 1 & 4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 1 & 4 & 1 \\ 0 & 1 & 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) From the row-reduced echelon form, the pivot columns indicate the vectors that form a basis of Col(A). Then a basis of Col(A) is given by

$$B = \{a_1, a_2, a_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(c) No, since not every column has a pivot.

(d) Since the basis obtained in (b) has 3 vectors, the rank of A is 3.

(e) From the rank theorem, we have

$$\text{rank}(A) + \dim(\text{Nul}(A)) = m,$$

where m = # of columns. Then

$$3 + \dim(\text{Nul}(A)) = 6$$

$$\therefore \dim(\text{Nul}(A)) = 3.$$

(f) Considering the basis $B = \{a_1, a_2, a_3\}$ and the reduced echelon form we have

$$a_1 = a_1$$

$$a_2 = a_2$$

$$a_3 = a_3$$

$$a_4 = a_1$$

$$a_5 = a_1 + a_2$$

$$a_6 = 4a_1 + 5a_2$$

(g) From the reduced echelon form we can choose x_4, x_5 and x_6 as the 3 free variables. we then have

$$x_1 = -x_4 - x_5 - 4x_6$$

$$x_2 = -x_5 - 5x_6$$

$$x_3 = 0$$

and therefore the general solution has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -x_4 - x_5 - 4x_6 \\ -x_5 - 5x_6 \\ 0 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -4 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where $x_4, x_5, x_6 \in \mathbb{R}$.

(h) Since the dimension of $\mathrm{Nul}(A) = 3$ we have 3 free variables in the general solution to $Ax = 0$.

(i) From the reduced echelon form we can choose $x_3 = u$, $x_4 = v$, $x_5 = w$ as parameters and write

$$x_1 = 1 - u - v - 4w$$

$$x_2 = 1 - v - 5w$$

$$x_3 = 1$$

$$x_4 = u$$

$$x_5 = v$$

$$x_6 = w$$

then the general solution can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -4 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where $u, v, w \in \mathbb{R}$.

(j) There are many vectors $c \in \mathbb{R}^4$ such that $Ax = c$ has no solution. The only condition needed here is that $c \notin \mathrm{Col}(A)$.

One such vector is

$$c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

2] We can write the system in matrix form $Ax=b$, with

$$A=\begin{bmatrix} 1 & 1 & 3 \\ -2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } b=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Note that $\det(A) = -3$ and the system has a unique solution. By Cramer's rule, we have

$$z = \frac{\det(A_3(b))}{\det(A)} = \frac{\det\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}}{\det(A)}$$

$$= \frac{8}{-3} = -\frac{8}{3}.$$

3) $W = \{ x_1 - x_2 + x_3 = 0 , x_2 + x_3 - x_4 = 0 \}$

a) W is a subspace of $\mathbb{R}^4$:

(i) $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in W$, since

$0 - 0 + 0 = 0$ and $0 + 0 - 0 = 0$.

(ii) Let $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in W$.

Then $u+v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ u_4 + v_4 \end{bmatrix}$. For $u+v \in W$ we need

$u_1 + v_1 - (u_2 + v_2) + u_3 + v_3 = 0$ and $(u_2 + v_2) + (u_3 + v_3) - (u_4 + v_4) = 0$

In fact,

$u_1 + v_1 - (u_2 + v_2) + u_3 + v_3 = u_1 - u_2 + u_3 + v_1 - v_2 + v_3 = 0$

and

$$(u_2+v_2)+(u_3+v_3)-(u_4+v_4) = u_2+u_3-u_4+v_2+v_3-v_4=0$$

Therefore $u+v \in W$.

(iii) For $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in W$ and $c \in \mathbb{R}$,

$$cu = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \\ cu_4 \end{bmatrix}$$ and

$$cu_1 - cu_2 + cu_3 = c(u_1 - u_2 + u_3) = 0,$$

$$cu_2 + cu_3 - cu_4 = c(u_2+u_3-u_4) = 0.$$

Therefore $cu \in W$.

From (i), (ii) and (iii) we conclude that W is a subspace of $\mathbb{R}^4$.

(b) We can think of W as the solution set of

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ x_2 + x_3 - x_4 &= 0 \end{aligned} \Longrightarrow \begin{aligned} x_1 &= x_2 - x_3 \\ x_4 &= x_2 + x_3 \end{aligned}$$

then, if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in W$ we have

$$x = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \\ x_2 + x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}.$$

then $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent (vectors not multiple of each other)

Therefore a basis of W is

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(c) From (b) we have that

$$\dim(W) = 2.$$

4) $T : \mathbb{R}^4 \longrightarrow \mathbb{P}_2$

$$x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longrightarrow T(x) = (a+b)t^2 + bt + c - d$$

(a) Let $x_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \in \mathbb{R}^4$

and $\alpha \in \mathbb{R}$. Then

(i) $T(x_1 + x_2) = (a_1 + a_2 + b_1 + b_2)t^2 + (b_1 + b_2)t + c_1 + c_2 - (d_1 + d_2)$

$= (a_1 + b_1)t^2 + b_1 t + c_1 - d_1 + (a_2 + b_2)t^2 + b_2 t + c_2 - d_2$

$= T(x_1) + T(x_2)$

(ii) $T(\alpha x_1) = (\alpha a_1 + \alpha b_1)t^2 + \alpha b_1 t + \alpha c_1 - \alpha d_1$

$= \alpha\left[(a_1 + b_1)t^2 + b_1 t + c_1 - d_1\right]$

$= \alpha T(x_1)$

From (i) and (ii) T is linear.

(b) $\text{Ker}(T) = \{ x \in \mathbb{R}^4 : T(x) = 0 \}$

For $x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \text{Ker}(T)$ we need

$T(x) = 0$, that is

$$(a+b)t^2 + bt + c - d = 0 \in \mathbb{P}_2 .$$

Then

$$\begin{aligned} a+b &= 0 \\ b &= 0 \\ c-d &= 0 \end{aligned} \implies \begin{aligned} a = b = 0 \\ c = d \end{aligned}$$

Therefore, if $x \in \text{Ker}(T)$, we have

$$x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c \\ c \end{bmatrix} = c \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad c \in \mathbb{R}.$$

and then

$$\text{Ker}(T) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(c) $\text{Range}(T) = \{ p \in \mathbb{P}_2 : T(x) = p \text{ for } x \in \mathbb{R}^4 \}$

Let $p(t) = At^2 + Bt + C \in \mathbb{P}_2$. If $p \in \text{Range}(T)$

there must exist $x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$ such that

$T(x) = p$. That is

$$(a+b)t^2 + bt + c-d = At^2 + Bt + C.$$

We then have the system

$$\begin{aligned} a+b &= A \\ b &= B \\ c-d &= C \end{aligned} \quad \text{or} \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & A \\ 0 & 1 & 0 & 0 & B \\ 0 & 0 & 1 & -1 & C \end{array}\right].$$

This system is always consistent, since there are pivots in each column. Therefore any $p \in \mathbb{P}_2$ is the image of some $x \in \mathbb{R}^4$ under T. From here,

$$\text{Range}(T) = \mathbb{P}_2.$$

5

(a) True.

If AB is invertible, then

$\det(AB) \neq 0$

$\Rightarrow \det(A) \cdot \det(B) \neq 0$

$\Rightarrow \det(A) \neq 0$ and $\det(B) \neq 0$.

Therefore A and B are both invertible.

(b) True.

Let $c_1, \dots, c_n$ be scalars such that

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0 \in V.$$

then

$$T(c_1 v_1 + \dots + c_n v_n) = T(0) = 0 \in U$$

T linear

$$\Rightarrow c_1 T(v_1) + \dots + c_n T(v_n) = 0 \in U$$

since $T(v_1), \dots, T(v_n)$ are l.i., we must have

$$c_1 = c_2 = \dots = c_n = 0$$

so $v_1, \dots, v_n$ are l.i.

(c) True.

Since $\mathcal{B}$ and $\mathcal{C}$ are bases of V, there exists a unique change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$, $\underset{C \leftarrow B}{P}$, whose columns are the coordinates of the vectors of $\mathcal{B}$ relative to $\mathcal{C}$.

$\underset{C \leftarrow B}{P}$ is $n \times n$ and also invertible.

From the IMT, the rank of $\underset{C \leftarrow B}{P}$ has to be n.