1) The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 2 \\ 2 & 4 & -3 & 4 & 5 \\ 5 & 10 & -8 & 11 & 12 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(1p) For obtaining an echelon form

From here we claim that the system has ∞-many sols (2 pivots and 4 variables), with 2 free parameters (# vars − # pivots).

If we set x_2 and x_4 as free parameters, we have

$$x_1 = 4 - 2x_2 + x_4$$

(1p) For the analysis

$$x_3 = 1 + 2x_4$$

and then, the general solution has the parametric vector form

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}.$$

(1p) For writing the solution in parametric vector form

2) The augmented matrix of $A\underline{x} = \underline{b}$ is

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 4 & 8 & k \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right]$$

From here, if

(i) $8-4h \neq 0$, that is $h \neq 2$, the system has a unique solution (pivots in every row)

1p

(ii) $8-4h = 0$ and $k-8=0$, that is

$$h = 2 \quad \text{and} \quad k = 8,$$

the system has ∞-many solutions (1 pivot and 2 variables), depending on 1 free variable.

1p

(iii) if $8-4h=0$ and $k-8\neq 0$,

the system is equivalent to

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 1 \end{array}\right],$$

which is inconsistent. Therefore, for $h=2$ and $k\neq 8$ the system has no solution.

1p

Remark: If the analysis is correct, but it comes from an incorrect row-reduced augmented matrix, give a max of 2.5p.

3

(a) True. The system $\left[\begin{array}{cc|c} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{array}\right]$

is consistent. In fact

$$\left[\begin{array}{cc|c} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 5/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{array}\right],$$

from where

$$\underline{u} = \frac{5}{2}\underline{v} + \frac{3}{2}\underline{w} \quad \text{and} \quad \underline{u} \in \operatorname{span}\{\underline{v}, \underline{w}\}.$$

1p for a correct argument.

(b) False. If such T exists, it must hold

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right),$$

but $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

1p for a correct argument

c) True. If v_1, v_2, v_3, v_4 are l.i., then

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \underline{0} \Rightarrow c_i = 0$$

for $i = 1, \dots, 4$ and c_i scalars.

In particular

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + 0 \cdot v_4 = 0$$

Also implies $c_1 = c_2 = c_3 = 0$. Therefore, $\{v_1, v_2, v_3\}$ is a linearly independent set.

(In general, any subset of a l.i. set is l.i.)

1p for a correct argument.