Vrije Universiteit Amsterdam, Faculty of Science **June 28, 2021**
Linear Algebra resit exam **12.15-14.45**
grade = points / 5 + 1

Use of calculators, books or notes is not allowed. Motivate your answers.

Question 1 (4p)

Given are the matrix $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & -3 \\ -2 & 6 & -1 \\ 3 & -8 & k \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ k \\ 8 \end{bmatrix}$.

Determine the value(s) of k (if any) for which
(a) the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent;
(b) the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;
(c) the columns of A are linearly independent;
(d) the columns of A span $\mathbb{R}^4$.

Question 2 (2p,2p)

Given is the matrix $B = \begin{bmatrix} 2 & 6 & -2 & -4 & 4 \\ 1 & -3 & 3 & 0 & 2 \\ -3 & -3 & 1 & 1 & 0 \end{bmatrix}$.

(a) Give the LU-factorization of the matrix B.
(b) Find a basis for the null space of B.

Question 3 (2p,2p,2p)

Given is the matrix $M = \begin{bmatrix} 4 & x \\ 4 & 0 \end{bmatrix}$ with $x \in \mathbb{R}$.

(a) For which value(s) of x does M have complex eigenvalues?

(b) Let $x = -5$. Find an invertible matrix P, and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $M = PCP^{-1}$.

(c) Is M diagonalizable if $x = -1$? Explain.

Question 4 (4p)

Give a Singular Value Decomposition of the matrix $H = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$.

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Question 5 (3p,2p,2p)

Let W be a subspace of $\mathbb{R}^4$ spanned by $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \\ 3 \end{bmatrix}$ and $\mathbf{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -4 \\ 2 \end{bmatrix}$.

(a) Find an orthogonal basis for W.

Let $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. The system $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{y}$ is inconsistent.

(b) Find the best approximation of $\mathbf{y}$ by vectors of W.
(c) Find the least-squares solution of the system $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{y}$.

Question 6 (12p)

Mark each of the following statements *true* or *false*. If the statement is *true*, give a proof. If the statement is *false*, give a proof or provide a counterexample.

(a) Let A be an $m \times n$ matrix. If $m < n$, then the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for all $\mathbf{b} \in \mathbb{R}^m$.

(b) Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$ be nonzero. The solution set of the inhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is a subspace of $\mathbb{R}^n$.

(c) Let A be an $n \times n$ matrix. If $A^{2021} = 0$, then A is not invertible.

(d) Let A be a 3×2 matrix and let the linear transformations $S : \mathbb{R}^2 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $S(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{y}) = A^T\mathbf{y}$. If S is one-to-one, then T is onto.

(e) If A and B are similar $n \times n$ matrices, then $\det A = \det B$.

(f) If $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are orthogonal, then $\mathbf{x}$ and $\mathbf{y}$ have equal length.

Question 7 (2p,2p,2p,2p)

Let Q be the set of polynomials of degree at most 2 that intersect the x-axis in $x = 1$. In set notation: $Q = \{\mathbf{p} \in \mathbb{P}_2 : \mathbf{p}(1) = 0\}$.

(a) Prove that Q is a subspace of $\mathbb{P}_2$.
(b) The dimension of Q is 2. Explain why $\mathbf{q_1}(x) = x - 1$ and $\mathbf{q_2}(x) = 5x^2 - x - 4$ form a basis for Q.

Let the inner product on $\mathbb{P}_2$ be defined by $\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(-1)\mathbf{q}(-1) + \mathbf{p}(0)\mathbf{q}(0) + \mathbf{p}(1)\mathbf{q}(1)$.

(c) Show that $\mathbf{q_1}(x) = x - 1$ and $\mathbf{q_2}(x) = 5x^2 - x - 4$ are orthogonal.
(d) Find the orthogonal projection of $\mathbf{p}(x) = 1 + 2x + x^2$ onto Q.

-END OF EXAM-