Linear Algebra

resit 2021

solutions

1a. $\left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & 2 & -3 & 1 \\ -2 & 6 & -1 & k \\ 3 & -8 & k & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & 2 & -3 & 1 \\ 0 & -2 & 3 & k+4 \\ 0 & 4 & k-6 & 2 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 0 & k+5 \\ 0 & 0 & k & 0 \end{array}\right]$

Consistent if $k = -5$.

1b. Unique solution if A has a pivot in every column ($k \neq 0$) <u>and</u> $A\underline{x} = \underline{b}$ is consistent, so if $k = -5$.

1c. The columns of A are linearly independent if A has a pivot in every column, so if $k \neq 0$.

1d. This is not possible: A only has three columns (not a pivot in every row).

2a. $$B = \begin{bmatrix} 2 & 6 & -2 & -4 & 4 \\ 1 & -3 & 3 & 0 & 2 \\ -3 & -3 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 6 & -2 & -4 & 4 \\ 0 & -6 & 4 & 2 & 0 \\ 0 & 6 & -2 & -5 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 6 & -2 & -4 & 4 \\ 0 & -6 & 4 & 2 & 0 \\ 0 & 0 & 2 & -3 & 6 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & -1 & 1 \end{bmatrix}$$

2b. $$U \sim \begin{bmatrix} 2 & 6 & 0 & -7 & 10 \\ 0 & -6 & 0 & 8 & -12 \\ 0 & 0 & 2 & -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & 1 & -2 \\ 0 & -6 & 0 & 8 & -12 \\ 0 & 0 & 1 & -3/2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1/2 & -1 \\ 0 & 1 & 0 & -4/3 & 2 \\ 0 & 0 & 1 & -3/2 & 3 \end{bmatrix} \longrightarrow \begin{cases} x_1 = -\frac{1}{2}x_4 + x_5 \\ x_2 = \frac{4}{3}x_4 - 2x_5 \\ x_3 = \frac{3}{2}x_4 - 3x_5 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases}$$

$$\underline{x} = x_4 \begin{bmatrix} -1/2 \\ 4/3 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Basis Nul B : $\left\{ \begin{bmatrix} -1/2 \\ 4/3 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

3a. $\det(M - \lambda I) = \begin{vmatrix} 4-\lambda & x \\ 4 & -\lambda \end{vmatrix} = -\lambda(4-\lambda) - 4x = 0$

$\lambda^2 - 4\lambda - 4x = 0$

$(\lambda - 2)^2 = 4 + 4x < 0$ if $x < -1$.

3b. Let $x = -5$. Then $(\lambda-2)^2 = -16$

$\lambda - 2 = \pm 4i$

$\lambda_1 = 2+4i, \ \lambda_2 = 2-4i$.

$$M - (2-4i)I = \begin{bmatrix} 2+4i & -5 \\ 4 & -2+4i \end{bmatrix} \sim \begin{bmatrix} 2+4i & -5 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow \underline{v}_1 = \begin{bmatrix} 5 \\ 2+4i \end{bmatrix}$$

$$P = \begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}, \quad C = P^{-1}MP = \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix}.$$

3c. Let $x = -1$. Then $(\lambda - 2)^2 = 0$, so $\lambda = 2$ (with multiplicity 2).

$$M - 2I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

This results in only one linearly independent eigenvector, so M is not diagonalizable if $x = -1$.

4. $\mathcal{H}^T\mathcal{H} = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$

$$\det(\mathcal{H}^T\mathcal{H} - \lambda I) = \begin{vmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{vmatrix} = (25-\lambda)^2 - 15^2 = 0$$

$$(25-\lambda)^2 = 15^2$$

$$25 - \lambda = \pm 15$$

$$\lambda_1 = 40, \quad \lambda_2 = 10.$$

$$\sigma_1 = \sqrt{40}, \quad \sigma_2 = \sqrt{10} \quad \rightarrow \quad \Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix}$$

$$\mathcal{H}^T\mathcal{H} - 40I = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underline{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\mathcal{H}^T\mathcal{H} - 10I = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\underline{u}_1 = \frac{1}{\sigma_1}\mathcal{H}\underline{v}_1 = \frac{1}{\sqrt{40}}\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4/\sqrt{80} \\ 8/\sqrt{80} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\underline{u}_2 = \frac{1}{\sigma_2}\mathcal{H}\underline{v}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4/\sqrt{20} \\ -2/\sqrt{20} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

$$\mathcal{U} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}. \qquad \mathcal{H} = \mathcal{U}\Sigma V^T.$$

5a. $\underline{v}_1 = \underline{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$$\underline{v}_2 = \underline{w}_2 - \frac{\underline{w}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -3 \\ 3 \end{bmatrix} - \frac{-8}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\underline{v}_3 = \underline{w}_3 - \frac{\underline{w}_3 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 - \frac{\underline{w}_3 \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2$$

$$= \begin{bmatrix} 2 \\ 0 \\ -4 \\ 2 \end{bmatrix} - \frac{-4}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} - \frac{8}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

orthogonal basis for W: $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$.

5b. $$\hat{\underline{y}} = \frac{\underline{y} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 + \frac{\underline{y} \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2 + \frac{\underline{y} \cdot \underline{v}_3}{\underline{v}_3 \cdot \underline{v}_3} \underline{v}_3$$

$$= \frac{-2}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \underline{0} + \frac{-4}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -1/2 \\ 1/2 \\ 3/2 \end{bmatrix}.$$

5c. Solve $c_1 \underline{w}_1 + c_2 \underline{w}_2 + c_3 \underline{w}_3 = \hat{\underline{y}}$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -3/2 \\ -1 & 1 & 0 & -1/2 \\ 1 & -3 & -4 & 1/2 \\ -1 & 3 & 2 & 3/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & -3/2 \\ 0 & 0 & 2 & -2 \\ 0 & -2 & -6 & 2 \\ 0 & 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & -3/2 \\ 0 & 0 & 2 & -2 \\ 0 & -2 & -6 & 2 \\ 0 & 0 & -2 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1/2 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \hat{\underline{c}} = \begin{bmatrix} 5/2 \\ 2 \\ -1 \end{bmatrix}.$$

6a. <u>False</u> If A has less rows than columns, then there can still be a row without a pivot, and the system can be inconsistent.

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$

6b. <u>False</u> The zero vector ($\underline{x} = \underline{0}$) is not an element of the solution set, so it is not a subspace.

6c. <u>True</u> If $A^{2021} = 0$, then $\det(A^{2021}) = 0$. and $\det(A^{2021}) = (\det A)^{2021}$, which implies $\det A = 0$. Therefore A is not invertible.

6d. <u>True</u> If S is one-to-one, then A must have a pivot in every column. Then A^T has a pivot in every row, and therefore T is onto.

6e. <u>True</u> If A and B are similar, then there exists an invertible matrix P such that $A = PBP^{-1}$. It follows that

$$\det A = \det(PBP^{-1}) = (\det P)(\det B)(\det P^{-1})$$
$$= \det B \cdot \det(PP^{-1}) = \det B \cdot \det I = \det B.$$

6f. <u>True</u> We know $(\underline{x}+\underline{y}) \cdot (\underline{x}-\underline{y}) = 0$. It follows that $\underline{x}\cdot\underline{x} + \underline{y}\cdot\underline{x} - \underline{x}\cdot\underline{y} - \underline{y}\cdot\underline{y} = \|\underline{x}\|^2 - \|\underline{y}\|^2 = 0$, so $\|\underline{x}\|^2 = \|\underline{y}\|^2$, and therefore $\|\underline{x}\| = \|\underline{y}\|$.

7a. Let $p, q \in Q$. This means $p(1) = 0$ and $q(1) = 0$. Then $(p+q)(1) = p(1) + q(1) = 0$, so $p+q \in Q$.

Let $p \in Q$ and $c \in \mathbb{R}$. This means $p(1) = 0$. Then $cp(1) = c \cdot 0 = 0$, so $cp \in Q$.

This proves that Q is a subspace of $\mathbb{P}_2$.

7b. Since $q_1(1) = 0$ and $q_2(1) = 0$, both are elements of Q.

and q_1 and q_2 are clearly linearly independent (q_2 has an x^2 and q_1 has not).

Since the dimension of Q is 2, q_1 and q_2 must form a basis for Q (by the Basis Theorem).

7c.

$$q_1(-1) = -2 \qquad q_2(-1) = 2$$
$$q_1(0) = -1 \qquad q_2(0) = -4$$
$$q_1(1) = 0 \qquad q_2(1) = 0$$

$$\langle q_1, q_2 \rangle = -2 \cdot 2 + -1 \cdot -4 + 0 \cdot 0 = 0.$$

7d. $\hat{p}(x) = \dfrac{\langle p, q_1\rangle}{\langle q_1, q_1\rangle} q_1 + \dfrac{\langle p, q_2\rangle}{\langle q_2, q_2\rangle} q_2$

$p(-1) = 0$
$p(0) = 1$
$p(1) = 4$

$$\hat{p}(x) = \frac{0 \cdot -2 + 1 \cdot -1 + 4 \cdot 0}{(-2)^2 + (-1)^2 + 0^2} q_1 + \frac{0 \cdot 2 + 1 \cdot -4 + 4 \cdot 0}{2^2 + (-4)^2 + 0^2} q_2$$

$$= -\frac{1}{5}(x-1) - \frac{4}{20}(5x^2 - x - 4)$$

$$= -\frac{1}{5}x + \frac{1}{5} - x^2 + \frac{1}{5}x + \frac{4}{5}$$

$$= 1 - x^2.$$