Vrije Universiteit Amsterdam, Faculty of Science — **May 21, 2021**
Linear Algebra partial exam 2 — **12.15-14.15**
grade = points / 4 + 1

Use of calculators, books or notes is not allowed. Motivate your answers.

Question 1 (2p,2p)

Given is the matrix $A = \begin{bmatrix} 3 & -4 \\ 5 & -1 \end{bmatrix}$.

(a) Find the eigenvalues of A.
(b) Find a basis for each eigenspace of A.

Question 2 (2p,2p,2p)

Given is the matrix $B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ with $x \in \mathbb{R}$.

(a) Compute the characteristic polynomial and find the eigenvalues of B.
(b) Find the value(s) of x, if any, for which the dimension of $\text{Nul}(A - 2I)$ equals the multiplicity of the eigenvalue $\lambda = 2$.
(c) Find the value(s) of x, if any, for which B is diagonalizable.

Question 3 (2p,2p,2p)

Given is the inconsistent system $C\mathbf{x} = \mathbf{y}$, with $C = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 0 \\ 2 & 1 & 1 \\ -1 & 2 & 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix}$.

(a) Compute the orthogonal projection of $\mathbf{y}$ onto the column space of C.
(b) Find the least-squares solution of the system $C\mathbf{x} = \mathbf{y}$.
(c) Compute the least-squares error.

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Question 4 (1p,2p,1p,3p,2p)

Given is the matrix $M = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

(a) Find the eigenvalues of MM^T.
(b) Prove the following statement:

Let A be an $m \times n$ matrix. If $\lambda \neq 0$ is an eigenvalue of AA^T, then λ is also an eigenvalue of A^TA.

(c) What does this tell you about the eigenvalues of M^TM?
(d) Find an orthogonal basis for each eigenspace of M^TM.
(e) Orthogonally diagonalize M^TM (i.e. find an orthogonal matrix P and a diagonal matrix D such that $M^TM = PDP^T$).

Question 5 (2p,2p,2p,2p)

Mark each of the following statements *true* or *false*. If the statement is *true*, give a proof. If the statement is *false*, give a proof or provide a counterexample.

(a) Let A be an $n \times n$ matrix. If A has a zero column, then $\lambda = 0$ is an eigenvalue of A.

(b) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$. The function $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_2 + u_2v_1$ is an inner product on $\mathbb{R}^2$.

(c) Let A be a symmetric matrix. Each vector $\mathbf{y}$ in $\mathbb{R}^n$ can be written in the form $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, with $\hat{\mathbf{y}}$ in Col A and $\mathbf{z}$ in Nul A.

(d) If A is orthogonally diagonalizable, then A^2 is also orthogonally diagonalizable.

Question 6 (3p)

Consider the inner product space $\mathbb{P}_3$ of all polynomials of degree at most 3, with inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}(1)\mathbf{q}(1) + \mathbf{p}(2)\mathbf{q}(2) + \mathbf{p}(3)\mathbf{q}(3) + \mathbf{p}(4)\mathbf{q}(4).$$

Find an orthogonal basis for the subspace spanned by $\mathbf{p}(t) = 1$, $\mathbf{q}(t) = 2t$ and $\mathbf{r}(t) = t^2 - 4t$.

-END OF EXAM-