Linear Algebra

Exam 2, 2021

Solutions

1a. $\begin{vmatrix} 3-\lambda & -4 \\ 5 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 20$

$$\lambda^2 - 2\lambda + 17 = 0$$

$$(\lambda - 1)^2 = -16$$

$$\lambda_1 = 1+4i, \quad \lambda_2 = 1-4i$$

1b. $A - (1+4i)I = \begin{bmatrix} 2-4i & -4 \\ 5 & -2-4i \end{bmatrix} \sim \begin{bmatrix} 1-2i & -2 \\ 0 & 0 \end{bmatrix}$

gives $\underline{v}_1 = \begin{bmatrix} 2 \\ 1-2i \end{bmatrix}$.

basis eigenspace $\lambda_1 = 1+4i$: $\left\{ \begin{bmatrix} 2 \\ 1-2i \end{bmatrix} \right\}$

basis eigenspace $\lambda_2 = 1-4i$: $\left\{ \begin{bmatrix} 2 \\ 1+2i \end{bmatrix} \right\}$ $(= \overline{\underline{v}_1})$

2a. $\begin{vmatrix} 1-\lambda & 0 & 0 & 2 \\ 0 & 2-\lambda & 0 & x \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 2 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & x \\ 0 & 0 & 2-\lambda \end{vmatrix}$

$= (1-\lambda)(1-\lambda)(2-\lambda)(2-\lambda) = 0$

$\lambda_1 = 1 \,, \quad \lambda_2 = 2$

(both multiplicity 2)

2b. $B - 2I = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & x \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

There must be two free variables. This is only the case for $x = 0$.

2c. Consider $B - I = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

There must be two free variables. But for any value of x, there is only one free variable. So, B is not diagonalizable for any value of x.

3a. Note that the columns of C are orthogonal. Therefore, we can use the formula

$$\hat{y} = \frac{y \cdot c_1}{c_1 \cdot c_1} c_1 + \frac{y \cdot c_2}{c_2 \cdot c_2} c_2 + \frac{y \cdot c_3}{c_3 \cdot c_3} c_3$$

$$= \frac{10}{10} c_1 + \frac{30}{10} c_2 + 0 c_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$$

3b. From part (a) we know $\hat{y} = 1 \cdot c_1 + 3 \cdot c_2 + 0 \cdot c_3$, so the least-squares solution is $\hat{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$.

Or solve the normal equations $C^T C \hat{x} = C^T y$:

$$C^T C = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad C^T y = \begin{bmatrix} 10 \\ 30 \\ 0 \end{bmatrix} \quad \Rightarrow \hat{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

3c. $$\|y - \hat{y}\| = \left\| \begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ -5 \\ 0 \\ 5 \end{bmatrix} \right\| = \sqrt{5^2 + 5^2} = \sqrt{50} \left(= 5\sqrt{2}\right).$$

$\left(= \|y - C\hat{x}\|\right)$

4a. $MM^T = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$. $\lambda_1 = 2$, $\lambda_2 = 4$.

4b. Let $\lambda \neq 0$ be an eigenvalue of AA^T, then

$AA^T \underline{x} = \lambda \underline{x}$ for some nonzero $\underline{x}$.

It follows that $A^T(AA^T \underline{x}) = A^T(\lambda \underline{x})$, so

$A^TA(A^T\underline{x}) = \lambda(A^T\underline{x})$, with $A^T\underline{x}$ nonzero.

Hence λ is an eigenvalue of A^TA.

4c. This means that the eigenvalues of M^TM are $\lambda_1 = 2$, $\lambda_2 = 4$ and (possibly) $\lambda_3 = 0$.

4d. $M^TM = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

$M^TM - 2I = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

Orthogonal basis eigenspace $\lambda_1 = 2$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

$$M^TM - 4I = \begin{bmatrix} -2 & 1 & 0 & -1 \\ 1 & -3 & 1 & -1 \\ 0 & 1 & -2 & -1 \\ -1 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Orthogonal basis eigenspace $\lambda_2 = 4$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

$$M^TM = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \underline{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \underline{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Note that $\underline{v}_3$ and $\underline{v}_4$ are not orthogonal.

Compute $\underline{x}_4 = \underline{v}_4 - \dfrac{\underline{v}_4 \cdot \underline{v}_3}{\underline{v}_3 \cdot \underline{v}_3} \underline{v}_3 = \underline{v}_4 - \dfrac{-2}{6} \underline{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1 \end{bmatrix}$.

Orthogonal basis eigenspace $\lambda_3 = 0$: $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

4e. Normalize the eigenvectors:

$$\underline{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \quad \underline{u}_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{bmatrix}, \quad \underline{u}_4 = \begin{bmatrix} 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ 3/\sqrt{12} \end{bmatrix}$$

$M^TM = PDP^T$ with

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/\sqrt{6} & 1/\sqrt{12} \\ 0 & 1/2 & -2/\sqrt{6} & 1/\sqrt{12} \\ -1/\sqrt{2} & 1/2 & 1/\sqrt{6} & 1/\sqrt{12} \\ 0 & -1/2 & 0 & 3/\sqrt{12} \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5a. True If A has a zero column, then A is not invertible, and by the Invertible Matrix Theorem, $\lambda = 0$ is an eigenvalue of A.

5b. False $\langle \underline{u}, \underline{u} \rangle = u_1 u_2 + u_2 u_1 = 2u_1 u_2 \not\geq 0$

for some $\underline{u} \in \mathbb{R}^2$.

(counterexample:

$\underline{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, then $\langle \underline{u}, \underline{u} \rangle = -4$.)

5c. True Col A is a subspace of $\mathbb{R}^n$.

By the Orthogonal Decomposition Theorem, each y in $\mathbb{R}^n$ can be written (uniquely) in the form $y = \hat{y} + \underline{z}$, with $\hat{y} \in \text{Col } A$ and $\underline{z} \in (\text{Col } A)^\perp$.

Note that $(\text{Col } A)^\perp = \text{Nul } A^T = \text{Nul } A$, since A is symmetric.

5d. <u>True</u> If A is orthogonally diagonalizable, then A is symmetric. So, $A^T = A$.

Consider $(A^2)^T = (A \cdot A)^T = A^T \cdot A^T = (A^T)^2 = A^2$.

So, A^2 is symmetric, which implies that A^2 is orthogonally diagonalizable.

6. Use Gram-Schmidt:

$v_1(t) = p(t) = 1$

$$v_2(t) = q(t) - \frac{\langle q, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$\langle q, v_1 \rangle = 2 \cdot 1 + 4 \cdot 1 + 6 \cdot 1 + 8 \cdot 1 = 20$

$\langle v_1, v_1 \rangle = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 4$

$$v_2(t) = 2t - \frac{20}{4} \cdot 1 = 2t - 5$$

$$v_3(t) = r(t) - \frac{\langle r, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle r, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$\langle r, v_1 \rangle = -3 \cdot 1 + -4 \cdot 1 + -3 \cdot 1 + 0 \cdot 1 = -10$

$\langle r, v_2 \rangle = -3 \cdot -3 + -4 \cdot -1 + -3 \cdot 1 + 0 \cdot 3 = 10$

$\langle v_2, v_2 \rangle = -3 \cdot -3 + -1 \cdot -1 + 1 \cdot 1 + 3 \cdot 3 = 20$

$$v_3(t) = t^2 - 4t - \frac{-10}{4} \cdot 1 - \frac{10}{20}(2t - 5)$$

$$= t^2 - 4t + \frac{5}{2} - t + \frac{5}{2}$$

$$= t^2 - 5t + 5.$$

orthogonal basis: $\{1, 2t-5, t^2 - 5t + 5\}$