1a. $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -4 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

A does not have a pivot in every row/column, so A is not invertible.
This means that 0 is an eigenvalue of A.

1b. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, so it is an eigenvector corresponding to $\lambda = 2$.

1c. $\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 2 & -1 & 2-\lambda \end{vmatrix} = (1-\lambda) \cdot \begin{vmatrix} 1-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1-\lambda \\ 2 & -1 \end{vmatrix}$

$$= (1-\lambda)^2 (2-\lambda) + (1-\lambda) - 1 - 2(1-\lambda)$$
$$= (\lambda^2 - 2\lambda + 1)(2-\lambda) - \lambda - 2 + 2\lambda$$
$$= -\lambda^3 + 2\lambda^2 - \lambda + 2\lambda^2 - 4\lambda + 2 - 2 + \lambda$$
$$= -\lambda^3 + 4\lambda^2 - 4\lambda = 0$$
$$-\lambda(\lambda^2 - 4\lambda + 4) = 0$$
$$-\lambda(\lambda - 2)^2 = 0$$
$$\lambda_1 = 0, \quad \lambda_2 = 2$$

1d. $\lambda_2 = 2$ has multiplicity 2, so A is only diagonalizable if the dimension of the corresponding eigenspace is 2 (i.e. two linearly independent eigenvectors).

We check:

$$A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

only one free variable, so only one linearly independent eigenvector.

A is not diagonalizable.

2. $$T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = 3 + 3t^2 \rightarrow \left[T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)\right]_C = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = -1 - t^2 \rightarrow \left[T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)\right]_C = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

So, $$M = \begin{bmatrix} 3 & -1 \\ -3 & 1 \\ 3 & -1 \end{bmatrix}.$$

3a. True We know $A\underline{v} = \underline{v}$ (for some nonzero $\underline{v}$), and A is invertible, so we get $A^{-1}(A\underline{v}) = A^{-1}\underline{v}$. Hence $A^{-1}\underline{v} = \underline{v}$, so 1 is an eigenvalue of A^{-1}.

3b. False (This means the matrix is diagonalizable.)

Counterexample:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible, but

has two linearly independent eigenvectors, since it has two distinct eigenvalues (on the main diagonal: 0 and 1).