Vrije Universiteit Amsterdam, Faculty of Science **March 24, 2021**
Linear Algebra partial exam 1 **15.15-17.15**
grade = points / 4 + 1

Use of calculators, books or notes is not allowed. Motivate your answers.

Question 1 (4p)

Given are the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & k \\ -9 & 6k & 3 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$.

Find the value(s) of k for which the system $A\mathbf{x} = \mathbf{b}$ has
1) no solution;
2) infinitely many solutions;
3) a unique solution.
Explain your answers.

Question 2 (3p,3p)

Given is the matrix $B = \begin{bmatrix} 3 & 6 & -3 \\ -1 & -2 & 1 \\ 2 & 5 & 0 \\ 3 & 5 & -5 \end{bmatrix}$.

(a) Are the columns of B linearly dependent? If so, give the linear dependence relation. If not, explain why.
(b) Find a basis for
1) the column space of B;
2) the row space of B;
3) the null space of B.

Question 3 (4p)

Given is the system of equations

$$\begin{aligned} x + y - z &= 0 \\ 3x - 2y + z &= 3 \\ x + 3y - 2z &= 0. \end{aligned}$$

Use Cramer's rule to find z.

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Question 4 (2p,1p,2p,1p)

Given are the matrices $A = \begin{bmatrix} 3 & 2 \\ -7 & -5 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -7 \\ -13 & 16 \end{bmatrix}$.

(a) Find the matrix B that solves $AB = C$.
(b) Explain why the columns of A form a basis of $\mathbb{R}^2$.

Consider the bases $\mathcal{A} = \left\{ \begin{bmatrix} 3 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 5 \\ -13 \end{bmatrix}, \begin{bmatrix} -7 \\ 16 \end{bmatrix} \right\}$ of $\mathbb{R}^2$.

(c) Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{C}$.
(d) How does this change-of-coordinates matrix relate to the matrix B?

Question 5 (2p,2p,2p,2p)

Mark each of the following statements *true* or *false*. If the statement is *true*, give a proof. If the statement is *false*, give a proof or provide a counterexample.

(a) Let A be an $n \times n$ matrix. If the columns of A are linearly independent, then the columns of A^2 span $\mathbb{R}^n$.

(b) Let A and B be $n \times n$ matrices, and let B be invertible. Then $\det(BAB^{-1}) = \det(A)$.

(c) The set $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq y \right\}$ is a subspace of $\mathbb{R}^2$.

(d) If the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$, defined by $T(\mathbf{x}) = A\mathbf{x}$, is onto, then the dimension of the kernel of T is 2.

Question 6 (3p,2p,1p,2p)

Given is the set $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a + b + c = 0 \right\}$.

(a) Prove that H is a subspace of $\mathbb{R}^3$.
(b) Find a basis of H.

Let the linear transformation $T : H \to \mathbb{P}_2$ be defined by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = a + bt + ct^2$.

(c) Is T onto? Explain.
(d) Find two polynomials that together span the range of T.

-END OF EXAM-