Linear Algebra
exam 1, 2021
Solutions

1. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & k & 1 \\ -9 & 6k & 3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & k & 1 \\ 0 & 6k & 12 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & k & 1 \\ 0 & 0 & 12-3k^2 & 6-3k \end{array}\right]$

no solution : $12-3k^2=0$ and $6-3k\neq 0$

so if $k=-2$

infinitely many solutions : $12-3k^2=0$ and $6-3k=0$

so if $k=2$

unique solution : $12-3k^2\neq 0$

so if $k\neq \pm 2$

2a. $B = \begin{bmatrix} 3 & 6 & -3 \\ -1 & -2 & 1 \\ 2 & 5 & 0 \\ 3 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Not a pivot in every column:
the columns are linearly dependent.

$\begin{cases} c_1 = 5c_3 \\ c_2 = -2c_3 \\ c_3 \text{ is free} \end{cases}$ Choose c_3: let $c_3=1$.

Dependence relation: $c_1\underline{b}_1 + c_2\underline{b}_2 + c_3\underline{b}_3 = \underline{0}$

$\Rightarrow 5\underline{b}_1 - 2\underline{b}_2 + \underline{b}_3 = \underline{0}$.

(or any nonzero multiple)

2b. $B \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, pivot in columns/rows 1 and 2:

$$\text{basis Col } B = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 5 \end{bmatrix} \right\}$$

$$\text{basis Row } B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Solve $B\underline{x} = \underline{0}$: $\quad \underline{x} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

$$\text{basis Nul } B = \left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}$$

3. $A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

$$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}$$

$$= (4-3) - (-6-1) - (9+2)$$

$$= 1 + 7 - 11 = -3$$

$$\det A_3(\underline{b}) = \begin{vmatrix} 1 & 1 & 0 \\ 3 & -2 & 3 \\ 1 & 3 & 0 \end{vmatrix} = -3 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -3(3-1) = -6$$

Cramer's rule: $z = \dfrac{\det A_3(\underline{b})}{\det A} = \dfrac{-6}{-3} = 2.$

4a. $AB = C \Rightarrow B = A^{-1}C$ (provided A^{-1} exists)

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} -5 & -2 \\ 7 & 3 \end{bmatrix} = \frac{1}{-15+14}\begin{bmatrix} -5 & -2 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -7 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 2 \\ -7 & -3 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -13 & 16 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 4 & 1 \end{bmatrix}$$

(Or row reduce $[A \mid C]$.)

4b. A is 2×2 and <u>invertible</u>, so by the Invertible Matrix Theorem the columns of A form a basis of $\mathbb{R}^2$.

(or: A has a pivot in every row and every column)

4c. $$\left[\begin{array}{cc|cc} 5 & -7 & 3 & 2 \\ -13 & 16 & -7 & -5 \end{array}\right] \sim \left[\begin{array}{cc|cc} 5 & -7 & 3 & 2 \\ 2 & -5 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 3 & -1 & 0 \\ 2 & -5 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 3 & -1 & 0 \\ 0 & -11 & 4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1/11 & 3/11 \\ 0 & 1 & -4/11 & -1/11 \end{array}\right]$$

$$\underbrace{\begin{array}{cc} 1/11 & 3/11 \\ -4/11 & -1/11 \end{array}}_{= \underset{C\leftarrow A}{P}}$$

4d. $\underset{C\leftarrow A}{P} = B^{-1}$ $\quad \left(B = \underset{A\leftarrow C}{P} \right)$

5a. <u>True</u> If the columns of A $(n \times n)$ are linearly independent, then A is invertible (by the IMT). Then A^2 (the product of two invertible matrices) is invertible. So, the columns of A^2 span $\mathbb{R}^n$ (by the IMT).

5b. <u>True</u>

$$\begin{aligned} \det(BAB^{-1}) &= (\det B)(\det A)(\det B^{-1}) \\ &= (\det B)(\det B^{-1})(\det A) \\ &= \det(BB^{-1}A) \\ &= \det(IA) \\ &= \det A \end{aligned}$$

or: use $\det B^{-1} = \dfrac{1}{\det B}$

5c. <u>False</u> W is <u>not</u> closed under scalar multiplication:

Take $\begin{bmatrix} x \\ y \end{bmatrix} \in W$, then $x \geq y$.

Let $c = -1$. Then $c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$,

but $-x \leq -y$, so $c \begin{bmatrix} x \\ y \end{bmatrix} \notin W$.

5d. <u>True</u> If T is onto, then $\dim \operatorname{Col} A = 3$.

Since $\dim \operatorname{Nul} A + \dim \operatorname{Col} A = 5$

it follows that $\dim \operatorname{Nul} A = 2$.

Therefore, $\dim \operatorname{Ker} T = \dim \operatorname{Nul} A = 2$.

<u>Or</u>: A is 3×5 and has a pivot in every row, so there are 2 free variables.

Therefore, $\dim \ker T = 2$.

6a. • If $a, b, c = 0$, then $a+b+c = 0$, so $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$.

• Take $\underline{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in H.

Then $a+b+c = 0$ and $x+y+z = 0$.

It follows that $\underline{u} + \underline{v} = \begin{bmatrix} a+x \\ b+y \\ c+z \end{bmatrix} \in H$,

since $(a+x) + (b+y) + (c+z) = (a+b+c) + (x+y+z) = 0$.

• Take $\underline{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in H$ and $k \in \mathbb{R}$.

Then $a+b+c = 0$. It follows that

$$k\underline{u} = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix} \in H, \text{ since } ka + kb + kc = k(a+b+c) = 0.$$

So, H is a subspace of $\mathbb{R}^3$.

<u>Or</u> show H is a span of vectors:

$a+b+c=0$ implies $c=-a-b$, so

$$\mathcal{H} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a+b+c=0 \right\}$$

$$= \left\{ \begin{bmatrix} a \\ b \\ -a-b \end{bmatrix} \in \mathbb{R}^3 \right\}$$

$$= \left\{ a\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad a,b \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

6b. See explanation above.

So, a basis of $\mathcal{H}$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

6c. T is <u>not</u> onto, because the range of T is not equal to $\mathbb{P}_2$: only polynomials $p(t) = a + bt + ct^2$ with $a+b+c=0$ are in Range T.

6d. $$T\left(\begin{bmatrix} a \\ b \\ -a-b \end{bmatrix} \right) = a + bt + (-a-b)t^2$$

$$= a(1-t^2) + b(t-t^2), \quad a,b \in \mathbb{R}$$

$$= \text{Span}\{1-t^2, t-t^2\}.$$