1.

$$\left[\begin{array}{ccc|c} 2 & 4 & 8 & -2 \\ 1 & 1 & 1 & 1 \\ 4 & 6 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 0 & -1 & -3 & 2 \\ 0 & -2 & -6 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 2x_3 + 3 \\ x_2 = -3x_3 - 2 \\ x_3 \ \text{free} \end{cases} \longrightarrow \quad \underline{x} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

$(x_3 \in \mathbb{R})$

2. $\left[\begin{array}{ccc|c} -1 & 5 & -3 & 2 \\ -1 & -3 & 1 & h \\ 4 & 0 & h & 2 \\ 0 & 2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 5 & -3 & 2 \\ 0 & -8 & 4 & h-2 \\ 0 & 20 & h-12 & 10 \\ 0 & 2 & -1 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} -1 & 5 & -3 & 2 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & h+2 \\ 0 & 0 & h-2 & 0 \end{array}\right]$

a. No solution if $h+2 \neq 0$, so if $h \neq -2$.

b. There must be a free variable, so $h = 2$, but then the system is inconsistent. So there cannot be infinitely many solutions.

c. There must be no free variables, so the columns are linearly independent if $h \neq 2$.

d. There are more rows than columns, so there cannot be a pivot in every row. Therefore, the columns of A do not span $\mathbb{R}^4$. (Or: three vectors cannot span a 4-dimensional space)

3a. <u>False</u>. A has more rows than columns, so there cannot be a pivot in every row. Therefore, $A\underline{x} = \underline{b}$ does not have a solution for all $\underline{b} \in \mathbb{R}^m$.
(Thm. 4)

Or give a counterexample:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Then $A\underline{x} = \underline{b}$ is inconsistent.

3b. <u>False</u>. $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) =$

$$\begin{bmatrix} cx + cy \\ 0 \\ c^2 xy \end{bmatrix} = c\begin{bmatrix} x + y \\ 0 \\ cxy \end{bmatrix} \neq cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

or show:

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + y_1 + y_2 \\ 0 \\ (x_1 + x_2)(y_1 + y_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + y_1 + x_2 + y_2 \\ 0 \\ x_1 y_1 + x_2 y_1 + x_1 y_2 + x_2 y_2 \end{bmatrix} \neq T\left(\begin{bmatrix} x_1 \\ y_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$