Faculty of Sciences	Resit Exam Linear Algebra
Vrije Universiteit Amsterdam	Friday, July 3, 2020 (13:30-16:15)

Please read the following instructions carefully: Use of a basic calculator is allowed. Provide an argument or calculation at every question. When you are finished, make photos of your solutions and upload them as **a single pdf file** to Canvas. Solutions received after 16:25 will not be accepted. Make sure the pdf is **clearly readable**. Unreadable solutions will not be accepted. Save your original solutions until your grade is known.

This exam consists of 6 questions and a total of 36 points can be obtained. The grade is calculated as (number of points + 4)/4.

Question 1 [8 pnt]. Let

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}.$$

a) [3 pnt] Determine a basis for the null space of A.

b) [3 pnt] Determine a basis for the column space of A.

The matrix A can be viewed as a linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$.

c) [1 pnt] Is this transformation one-to-one? *Hint:* Use your answer to a).

d) [1 pnt] Is this transformation onto $\mathbb{R}^3$? *Hint:* Use your answer to b).

Question 2 [4 pnt]. We consider some basic properties of invertible matrices.

a) [1 pnt] Which relation must the entries of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfy in order for A to be invertible?

b) [1 pnt] Suppose that $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$.

c) [1 pnt] Suppose that A, B, and C are invertible $n \times n$ matrices. Show that ABC is invertible by constructing a matrix D such that $(ABC)D = I$ and $D(ABC) = I$, where I denotes the $n \times n$ identity matrix.

d) [1 pnt] Solve the equation $AB = BC$ for A, assuming that A, B and C are square and B is invertible.

Question 3 [7 pnt]. Consider the following matrix:

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

a) [1 pnt] Determine the characteristic polynomial of A.

b) [1 pnt] Show that $\lambda = 5$ and $\lambda = -3$ are eigenvalues of A.

c) [3 pnt] Determine a basis for the eigenspaces of $\lambda = 5$ and $\lambda = -3$.

d) [2 pnt] Find a matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (i.e. diagonalize A).

Question 4 [6 pnt]. Let E be the space spanned by the vectors $u_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

a) [1 pnt] What is the dimension of E?

b) [2 pnt] Determine an orthogonal basis for E.

c) [1 pnt] A basis for E is given by $\{w_1, w_2\}$ with $w_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $w_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. Show that $\{w_1, w_2\}$ is an orthonormal set.

d) [2 pnt] Calculate the orthogonal projection of $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto E.

Question 5 [5 pnt] A *subspace* V of $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ such that $0 \in V$ and if $x, y \in V$, then $c_1x + c_2y \in V$ for all $c_1, c_2 \in \mathbb{R}$.

a) [2 pnt] Let $x \in \mathbb{R}^3$. Show that the set $l = \{cx | c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

b) [2 pnt] Show that the orthogonal complement of a two-dimensional subspace of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

c) [1 pnt] What is its dimension?

Question 6 [6 pnt] Determine if the following statements are true or false. Provide an argument for your answer.

a) [1 pnt] For every two symmetric matrices A and B, AB is symmetric as well.

b) [1 pnt] The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

c) [1 pnt] A linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ with $n < m$ cannot be onto $\mathbb{R}^m$.

d) [1 pnt] If $\lambda + 3$ is a factor of the characteristic polynomial of A, then 3 is an eigenvalue of A.

e) [1 pnt] The null space of an $n \times m$ matrix is a subspace of $\mathbb{R}^n$.

f) [1 pnt] If A^2 is the zero matrix, then the only eigenvalue of A is 0.