Resit LA 2020

1a) $\begin{bmatrix} 2 & -2 & -2 & 2 & 2 \\ -4 & 0 & 4 & 6 & 0 \\ 1 & 1 & -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & -2 & 2 & 2 \\ 0 & \boxed{-4} & 0 & 10 & 4 \\ 0 & 2 & 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & -2 & 2 & 2 \\ 0 & -4 & 0 & 10 & 4 \\ 0 & 0 & 0 & \boxed{3} & 0 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix}.$

1b) A has a pivot in every row, so $A\underline{x} = \underline{b}$ is always consistent: for all $k \in \mathbb{R}$.

1c) $U \sim \begin{bmatrix} 2 & -2 & -2 & 0 & 2 \\ 0 & -4 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\begin{cases} x_1 = x_3 \\ x_2 = x_5 \\ x_3 \text{ is free} \\ x_4 = 0 \\ x_5 \text{ is free} \end{cases}$

$\underline{x} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

basis Nul A $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

1d) Yes, A has three pivot columns / rows.
(so the columns span $\mathbb{R}^3$).

2a) $\det B = \begin{vmatrix} -1 & h & 0 \\ 1 & 0 & -1 \\ -h & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} - h \begin{vmatrix} 1 & -1 \\ -h & 0 \end{vmatrix}$

$= -1(0+1) - h(0-h) = h^2 - 1.$

2b) $x_2 = \dfrac{\det B_2(\underline{d})}{\det B}$

$\det B_2(d) = \begin{vmatrix} -1 & 1 & 0 \\ 1 & 1 & -1 \\ -h & -1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ -h & 0 \end{vmatrix}$

$= -1(0-1) - 1(0-h) = h+1$

$x_2 = \dfrac{h+1}{h^2-1}$

2c) For $h \neq \pm 1$, then $\det B \neq 0$ so B is invertible.
(By the IMT, $B\underline{x} = \underline{y}$ is consistent for all $\underline{y} \in \mathbb{R}^3$.)

2d) $B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$; $\begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -\lambda & -1 \\ -1 & 1 & -\lambda \end{vmatrix} = (-1-\lambda) \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ -1 & -\lambda \end{vmatrix}$

$= (-1-\lambda)(\lambda^2+1) - (-\lambda-1)$

$= -\lambda^2 - \lambda^3 - 1 - \lambda + \lambda + 1$

$= -\lambda^2(1+\lambda) = 0$

$\lambda = 0 \quad \cup \quad \lambda = -1$

(multiplicity 2) (multiplicity 1)

2e) Check eigenspace of $\lambda = 0$:

$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ only 1 free variable, so eigenspace has dimension 1.

This is unequal to the multiplicity of $\lambda = 0$.
So B is not diagonalizable.

3a) $\underline{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix}$

$$\underline{v}_2 = \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix} - \frac{18}{36} \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\underline{v}_3 = \underline{x}_3 - \frac{\underline{x}_3 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 - \frac{\underline{x}_3 \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2$$

$$= \begin{bmatrix} 2 \\ 5 \\ 4 \\ -3 \end{bmatrix} - \frac{36}{36} \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 4 \\ -3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

orthogonal basis Col C $= \left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

3b) $Q = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 0 & 1/3 \\ 0 & 2/3 & 2/3 \\ -2/3 & 1/3 & 0 \end{bmatrix}$ $\quad (\|\underline{v}_1\| = 6, \ \|\underline{v}_2\| = 3, \ \|\underline{v}_3\| = 3)$

$$C = QR \Rightarrow R = Q^T C = \frac{1}{3} \begin{bmatrix} 1 & 2 & 0 & -2 \\ 2 & 0 & 2 & 1 \\ -2 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 \\ 4 & 2 & 5 \\ 0 & 2 & 4 \\ -4 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 & 6 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix}.$$

3c) $$\hat{\underline{y}} = \frac{\underline{y} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 + \frac{\underline{y} \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2 + \frac{\underline{y} \cdot \underline{v}_3}{\underline{v}_3 \cdot \underline{v}_3} \underline{v}_3$$

$$= \frac{-4}{36} \begin{bmatrix} 2 \\ 4 \\ 0 \\ -4 \end{bmatrix} + \frac{2}{9} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/9 \\ -4/9 \\ 0 \\ +4/9 \end{bmatrix} + \begin{bmatrix} 4/9 \\ 0 \\ 4/9 \\ 2/9 \end{bmatrix} + \begin{bmatrix} -2/9 \\ 1/9 \\ 2/9 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

4) $X = \begin{bmatrix} 1 & -4 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad \underset{\sim}{y} = \begin{bmatrix} -2 \\ -1 \\ 4 \\ 6 \end{bmatrix}, \quad \underset{\sim}{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}.$

$$X^T X \hat{\underset{\sim}{\beta}} = X^T \underset{\sim}{y}$$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & 21 \end{bmatrix}$$

$$X^T \underset{\sim}{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & -3 & 7 \\ -3 & 21 & 21 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 25 & 35 \\ 1 & -7 & -7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -7 & -7 \\ 0 & 1 & 7/5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 14/5 \\ 0 & 1 & 7/5 \end{array}\right]. \qquad y = \frac{14}{5} + \frac{7}{5}x.$$

5a) <u>True</u>. Suppose $A\underline{x} = \underline{b}$ has at most one solution for all $\underline{b} \in \mathbb{R}^m$. Then $A\underline{x} = \underline{0}$ has at most one solution: only the trivial solution $\underline{x} = \underline{0}$. So the columns of A are linearly independent.

5b) <u>False</u>. $T(\underline{0}) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \neq \underline{0}$. So T is not a linear transformation.

5c) <u>True</u>. If $A^T A$ is invertible, then $\det A^T A \neq 0$.

$\det A^T A = (\det A^T)(\det A) \neq 0$, so $\det A \neq 0$.

Therefore, A is invertible.

5d) <u>False</u>. Suppose $A\underline{v} = \lambda \underline{v}$ with $\lambda = a \pm bi$.

Then $A(A\underline{v}) = A(\lambda \underline{v})$

$A^2 \underline{v} = \lambda (A\underline{v}) = \lambda^2 \underline{v}$,

so λ^2 is an eigenvalue of A^2.

So if $a = 0$, that is, $\lambda = bi$, then $\lambda^2 = (bi)^2 = -b^2$ is not complex.

5e) <u>True</u>. $\|\underline{v}_1 + \underline{v}_2 + \underline{v}_3\| = \sqrt{\langle \underline{v}_1 + \underline{v}_2 + \underline{v}_3, \underline{v}_1 + \underline{v}_2 + \underline{v}_3 \rangle}$

$= \sqrt{\langle \underline{v}_1, \underline{v}_1 \rangle + \langle \underline{v}_2, \underline{v}_2 \rangle + \langle \underline{v}_3, \underline{v}_3 \rangle + 2\langle \underline{v}_1, \underline{v}_2 \rangle + 2\langle \underline{v}_1, \underline{v}_3 \rangle + 2\langle \underline{v}_2, \underline{v}_3 \rangle}$

$= \sqrt{\|\underline{v}_1\|^2 + \|\underline{v}_2\|^2 + \|\underline{v}_3\|^2 + 0 + 0 + 0} = \sqrt{1+1+1} = \sqrt{3}$.

5f) <u>True</u> AB is similar to BA if there exists an invertible matrix P such that $AB = PBAP^{-1}$.

Take $P = A$, then $AB = ABAA^{-1}$ is true.

(this works since A is invertible)

6a) No, because $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$.

6b) Yes, $\mathbb{P}_2$ is a subset of $\mathbb{P}_3$ and is closed under addition and scalar multiplication.

$(\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ and

$\mathbb{P}_3 = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.)$

6c)
$$\left.\begin{aligned} \langle 1, t\rangle &= 1\cdot -2 + 1\cdot 0 + 1\cdot 2 = 0 \\ \langle 1, 3t^2-8\rangle &= 1\cdot 4 + 1\cdot -8 + 1\cdot 4 = 0 \\ \langle t, 3t^2-8\rangle &= -2\cdot 4 + 0\cdot -8 + 2\cdot 4 = 0 \end{aligned}\right\}$$
polynomials are <u>orthogonal</u> and hence linearly independent.

Polynomials <u>span</u> $\mathbb{P}_2$: $\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has 3 pivots.

6d) Length $p(t) = 3t^2 - 8$:

$$\|p(t)\| = \sqrt{\langle p(t), p(t)\rangle} = \sqrt{4^2 + (-8)^2 + 4^2} = \sqrt{96} = 4\sqrt{6}.$$

7) $M^T M = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 6 \\ 6 & 4 \end{bmatrix}.$

$$\begin{vmatrix} 13-\lambda & 6 \\ 6 & 4-\lambda \end{vmatrix} = (13-\lambda)(4-\lambda) - 36 = 0$$
$$\lambda^2 - 17\lambda + 16 = 0$$
$$(\lambda - 16)(\lambda - 1) = 0 \quad \rightarrow \quad \lambda_1 = 16, \; \lambda_2 = 1$$
$$\sigma_1 = 4, \quad \sigma_2 = 1$$

$\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$

$$\left.\begin{aligned} \lambda_1 = 16 &: \begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \lambda_2 = 1 &: \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{aligned}\right\} \quad V = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}.$$

$$\left.\begin{aligned} \underline{u}_1 &= \frac{1}{\sigma_1} M \underline{v}_1 = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \\ \underline{u}_2 &= \frac{1}{\sigma_2} M \underline{v}_2 = \frac{1}{1}\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \end{aligned}\right\} \quad U = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}.$$