Vrije Universiteit Amsterdam, Faculty of Science | **May 29, 2020**
Linear Algebra final exam | **15.30-18.15**
grade = points / 5 + 1

Use of calculators, books or notes is not allowed. Motivate your answers.

Question 1 (2p,1p,2p)

Given are the matrix $A = \begin{bmatrix} 3 & 4 & 3 & -4 \\ 2 & 3 & 3 & -3 \\ 4 & 7 & 9 & -7 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix}$.

(a) Determine the value(s) of k for which the system $A\mathbf{x} = \mathbf{b}$ is consistent.
(b) Do the columns of A span $\mathbb{R}^3$? Explain.
(c) Find a basis for the null space of A.

Question 2 (2p,3p,3p,1p)

Given is the matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

(a) Compute the determinant of B.
(b) Find the eigenvalues of B.
(c) Give a basis for each eigenspace of B.
(d) Is B diagonalizable? Explain.

Question 3 (3p,2p,2p,2p)

Let W be a subspace of $\mathbb{R}^4$ spanned by $\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{w}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 8 \end{bmatrix}$.

(a) Find an orthogonal basis for W.

Let $\mathbf{y} = \begin{bmatrix} 3 \\ -8 \\ 9 \\ 0 \end{bmatrix}$. The system $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{y}$ is inconsistent.

(b) Find the best approximation of $\mathbf{y}$ by vectors of W.
(c) Find the least-squares solution of the system $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{y}$.
(d) Compute the least-squares error. (You do not need to simplify your answer.)

-EXAM CONTINUES ON THE BACK-

Question 4 (2p,2p,2p,2p,2p,2p)

Mark each of the following statements *true* or *false*. If the statement is *true*, give a proof. If the statement is *false*, give a proof or provide a counterexample.

(a) Let A be an $n \times n$ matrix with nonzero columns. If the columns of A are orthogonal, then the determinant of A is nonzero.

(b) There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 6 \\ 6 \end{bmatrix}$.

(c) The set $H = \left\{ \begin{bmatrix} a-2b \\ 3a \\ 2a+3b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

(d) If the matrix A is diagonalizable, then A is invertible.

(e) If the matrix A is invertible, then A is diagonalizable.

(f) Let A be an $n \times n$ matrix. If $A = A^2$ and λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$.

Question 5 (2p,2p,2p)

Let the transformation $T : \mathbb{R}^3 \to \mathbb{P}_2$ be defined by $T\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}\right) = (3a_0 + a_2)t + a_1 t^2$.

(a) Determine the kernel of T.

(b) Is T onto? Is T one-to-one? Explain.

(c) Find the matrix for T relative to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}$ for $\mathbb{R}^3$ and the standard basis $\mathcal{C} = \{1, t, t^2\}$ for $\mathbb{P}_2$.

Question 6 (3p,1p)

Given is the matrix $M = \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 4 & -2 \end{bmatrix}$.

(a) Give a Singular Value Decomposition of M.
(b) Now write down a Singular Value Decomposition of M^T.

-END OF EXAM-