Linear Algebra Final exam 2020 (May 29)

1a) $\left[\begin{array}{cccc|c} 3 & 4 & 3 & -4 & 1 \\ 2 & 3 & 3 & -3 & 2 \\ 4 & 7 & 9 & -7 & k \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 3 & -1 & 4 \\ 0 & 1 & 3 & -1 & k-4 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 3 & -1 & 4 \\ 0 & 0 & 0 & 0 & k-8 \end{array}\right]$ Consistent if $k=8$.

1b) No, not a pivot in every row / 2 pivots not 3 / not consistent for all $\underline{b} \in \mathbb{R}^3$.

1c) $\sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{cases} x_1 = 3x_3 \\ x_2 = -3x_3 + x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$, $\underline{x} = x_3 \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Basis Nul A: $\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

2a) $\det B = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0.$

or: $\det B = 1 \cdot \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$

$= 1 \cdot (6-6) - 2(3-3) + 3(2-2) = 0.$

2b) $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 1 & 2-\lambda & 3 \\ 1 & 2 & 3-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & 3 \\ 2 & 3-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 \\ 1 & 3-\lambda \end{vmatrix} + 3 \begin{vmatrix} 1 & 2-\lambda \\ 1 & 2 \end{vmatrix}$

$= (1-\lambda)\big((2-\lambda)(3-\lambda) - 6\big) - 2(3-\lambda-3) + 3(2-2+\lambda)$

$= (1-\lambda)(\lambda^2 - 5\lambda) + 2\lambda + 3\lambda$

$= \lambda^2 - 5\lambda - \lambda^3 + 5\lambda^2 + 5\lambda$

$= -\lambda^3 + 6\lambda^2$

$= -\lambda^2(\lambda - 6) = 0$

$\lambda = 0 \lor \lambda = 6.$

2c) $\lambda = 0$: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{cases} x_1 = -2x_2 - 3x_3 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases}$.

$$\underline{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Basis eigenspace for $\lambda = 0$: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\lambda = 6$: $\begin{bmatrix} -5 & 2 & 3 \\ 1 & -4 & 3 \\ 1 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 \\ 0 & -18 & 18 \\ 0 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, $\underline{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Basis eigenspace for $\lambda = 6$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

2d) Yes; B has three linearly independent eigenvectors/ the sum of the dimensions of the bases of the eigenspaces is three/ $B = PDP^{-1}$ with $P = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ <u>invertible</u> and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

3a) $\underline{x}_1 = \underline{w}_1$

$$\underline{x}_2 = \underline{w}_2 - \frac{\underline{w}_2 \cdot \underline{x}_1}{\underline{x}_1 \cdot \underline{x}_1} \underline{x}_1 = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 0 \end{bmatrix} - \frac{6}{6} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

$$\underline{x}_3 = \underline{w}_3 - \frac{\underline{w}_3 \cdot \underline{x}_1}{\underline{x}_1 \cdot \underline{x}_1} \underline{x}_1 - \frac{\underline{w}_3 \cdot \underline{x}_2}{\underline{x}_2 \cdot \underline{x}_2} \underline{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 8 \end{bmatrix} - \frac{-6}{6} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{20}{20} \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 6 \end{bmatrix}$$

orthogonal basis for W: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 6 \end{bmatrix} \right\}$.

3b) $$\hat{\underline{y}} = \frac{\underline{y} \cdot \underline{x}_1}{\underline{x}_1 \cdot \underline{x}_1} \underline{x}_1 + \frac{\underline{y} \cdot \underline{x}_2}{\underline{x}_2 \cdot \underline{x}_2} \underline{x}_2 + \frac{\underline{y} \cdot \underline{x}_3}{\underline{x}_3 \cdot \underline{x}_3} \underline{x}_3$$

$$= \frac{-2}{6} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \underline{0} + \underline{0} = \begin{bmatrix} -2/3 \\ -1/3 \\ 0 \\ 1/3 \end{bmatrix}$$

3c) Solve $\left[\begin{array}{ccc|c} 2 & 1 & 0 & -2/3 \\ 1 & 4 & 2 & -1/3 \\ 0 & 3 & 2 & 0 \\ -1 & 0 & 8 & 1/3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & -7 & -4 & 0 \\ 1 & 4 & 2 & -1/3 \\ 0 & 3 & 2 & 0 \\ 0 & 4 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

least-squares solution: $\hat{\underline{c}} = \begin{bmatrix} -1/3 \\ 0 \\ 0 \end{bmatrix}$

or: put vectors in a matrix W and compute

$$W^T W = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 4 & 3 & 0 \\ 0 & 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 3 & 2 \\ -1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -6 \\ 6 & 26 & 14 \\ -6 & 14 & 72 \end{bmatrix}$$

$$W^T \underline{y} = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 4 & 3 & 0 \\ 0 & 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ -8 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

and solve $\left[\begin{array}{ccc|c} 6 & 6 & -6 & -2 \\ 6 & 26 & 14 & -2 \\ -6 & 14 & 72 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 6 & 6 & -6 & -2 \\ 0 & 20 & 20 & 0 \\ 0 & 20 & 66 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -1/3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$, so $\hat{\underline{c}} = \begin{bmatrix} -1/3 \\ 0 \\ 0 \end{bmatrix}$.

3d) least-squares error:

$$\| \underline{y} - \hat{\underline{y}} \| = \left\| \begin{bmatrix} 3 \\ -8 \\ 9 \\ 0 \end{bmatrix} - \begin{bmatrix} -2/3 \\ -1/3 \\ 0 \\ 1/3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 11/3 \\ -23/3 \\ 9 \\ -1/3 \end{bmatrix} \right\|$$

$$= \sqrt{(11/3)^2 + (-23/3)^2 + 9^2 + (-1/3)^2}.$$

$$\left(= \sqrt{\frac{121}{9} + \frac{529}{9} + 81 + \frac{1}{9}} = \sqrt{\frac{1380}{9}} = \frac{2}{3}\sqrt{345} \right)$$

4a) True Orthogonal columns are linearly independent. A is square, so by the IMT, A is invertible and $\det A \neq 0$.

4b) False $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$ and $T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$, so $T\begin{pmatrix}2\\3\end{pmatrix} = T\begin{pmatrix}1\\1\end{pmatrix} + T\begin{pmatrix}1\\2\end{pmatrix}$
$= \begin{pmatrix}7\\6\end{pmatrix} \neq \begin{pmatrix}6\\6\end{pmatrix}$.

4c) True $\begin{bmatrix} a-2b \\ 3a \\ 2a+3b \end{bmatrix} = a\begin{bmatrix}1\\3\\2\end{bmatrix} + b\begin{bmatrix}-2\\0\\3\end{bmatrix}$, so $H = \mathrm{Span}\left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}-2\\0\\3\end{bmatrix}\right\}$

and any span of vectors is a subspace.

or: H is closed under addition:

Take $\underline{v}, \underline{u} \in H$. Let $v = \begin{bmatrix} a-2b \\ 3a \\ 2a+3b \end{bmatrix}$ and $u = \begin{bmatrix} c-2d \\ 3c \\ 2c+3d \end{bmatrix}$; $a,b,c,d \in \mathbb{R}$.

Then $\underline{v}+\underline{u} = \begin{bmatrix} a-2b+c-2d \\ 3a+3c \\ 2a+3b+2c+3d \end{bmatrix} = \begin{bmatrix} (a+c)-2(b+d) \\ 3(a+c) \\ 2(a+c)+3(b+d) \end{bmatrix} \in H$.

H is closed under scalar multiplication:

Take $v \in H$ and $k \in \mathbb{R}$.

Then $k\underline{v} = \begin{bmatrix} k(a-2b) \\ k(3a) \\ k(2a+3b) \end{bmatrix} = \begin{bmatrix} ka-2kb \\ 3ka \\ 2ka+3kb \end{bmatrix} \in H$.

Therefore, H is a subspace.

4d) False Counterexample: The matrix from question 2 is not invertible, but this matrix is diagonalizable.

other counter example: $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ has two distinct eigenvalues but determinant is zero.

4e) False Counterexample: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is invertible ($\det \neq 0$), but only one linearly independent eigenvector.

4f) True We know $A\underline{v} = \lambda\underline{v}$ for some $\underline{v} \neq \underline{0}$.

Then $A(A\underline{v}) = A(\lambda\underline{v})$

so $A^2\underline{v} = \lambda(A\underline{v}) = \lambda(\lambda\underline{v}) = \lambda^2\underline{v}$.

But $A = A^2$, so it follows that $A\underline{v} = \lambda^2\underline{v}$.

But we started with $A\underline{v} = \lambda\underline{v}$, so $\lambda\underline{v} = \lambda^2\underline{v}$.

Hence $(\lambda - \lambda^2)\underline{v} = \underline{0}$ so $\lambda(1-\lambda) = 0$.

Therefore, $\lambda = 0$ or $\lambda = 1$.

5a) $T\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = (3a_0 + a_2)t + a_1 t^2 = 0$

$\Longrightarrow \quad 3a_0 + a_2 = 0 \quad$ and $\quad a_1 = 0$

$a_0 = -\frac{1}{3} a_2 \quad$ and $\quad a_1 = 0 \quad$ and $\quad a_2$ is free.

$$\operatorname{Ker}(T) = \left\{ a_2 \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} : a_2 \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \right\}$$

5b) T is not onto : $p(t) = 3$ is not in the range of T, for example. / only polynomials $p(t) = b_0 + b_1 t + b_2 t^2$ with $b_0 = 0$ are reached.

T is not one-to-one: $\underline{0}$ is not the only element in the kernel of T. / $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3t$ but also $T\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = 3t$, for example.

5c) $$M = \left[[T(\underline{b}_1)]_C \; [T(\underline{b}_2)]_C \; [T(\underline{b}_3)]_C \right]$$

$$= \left[\left[T\left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}\right)\right]_C \; \left[T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)\right]_C \; \left[T\left(\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}\right)\right]_C \right]$$

$$= \left[[3t + 3t^2]_C \; [t + t^2]_C \; [8t]_C \right]$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 1 & 8 \\ 3 & 1 & 0 \end{bmatrix}.$$

6a) $M^T M = \begin{bmatrix} 4 & 1 & 4 \\ -2 & 5 & -2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 33 & -11 \\ -11 & 33 \end{bmatrix}$

eigenvalues:

$$\begin{vmatrix} 33-\lambda & -11 \\ -11 & 33-\lambda \end{vmatrix} = (33-\lambda)^2 - 11^2 = 0$$
$$(33-\lambda)^2 = 11^2$$
$$33-\lambda = 11 \quad \vee \quad 33-\lambda = -11$$
$$\lambda_2 = 22 \quad \vee \quad \lambda_1 = 44.$$

singular values: $\sigma_1 = \sqrt{44}$, $\sigma_2 = \sqrt{22}$

$$\Sigma = \begin{bmatrix} \sqrt{44} & 0 \\ 0 & \sqrt{22} \\ 0 & 0 \end{bmatrix}$$

eigenvectors:

$$\left.\begin{aligned} \lambda_1 = 44 &: \begin{bmatrix} -11 & -11 \\ -11 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \underline{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \lambda_2 = 22 &: \begin{bmatrix} 11 & -11 \\ -11 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow \underline{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}\right\} \text{normalize}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Compute U:

$$\frac{1}{\sigma_1} M\underline{v}_1 = \frac{1}{\sqrt{44}} \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{44}} \begin{bmatrix} 6/\sqrt{2} \\ -4/\sqrt{2} \\ 6/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 6/\sqrt{88} \\ -4/\sqrt{88} \\ 6/\sqrt{88} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{22} \\ -2/\sqrt{22} \\ 3/\sqrt{22} \end{bmatrix}$$

$$\frac{1}{\sigma_2} M\underline{v}_2 = \frac{1}{\sqrt{22}} \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{22}} \begin{bmatrix} 2/\sqrt{2} \\ 6/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{44} \\ 6/\sqrt{44} \\ 2/\sqrt{44} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{11} \\ 3/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}$$

Find third orthonormal vector.

Consider $\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. By inspection: take $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(or solve system $\underline{x} \cdot \underline{u}_1 = 0$, $\underline{x} \cdot \underline{u}_2 = 0$)

$$U = \begin{bmatrix} 3/\sqrt{22} & 1/\sqrt{11} & 1/\sqrt{2} \\ -2/\sqrt{22} & 3/\sqrt{11} & 0 \\ 3/\sqrt{22} & 1/\sqrt{11} & -1/\sqrt{2} \end{bmatrix}.$$

SVD: $M = U \Sigma V^T$

6b) $M^T = (U \Sigma V^T)^T = (V^T)^T \Sigma^T U^T = V \Sigma^T U^T.$