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| Faculty of Sciences | Testexam Linear Algebra |
| Vrije Universiteit Amsterdam | Wednesday, May 27, 2020 (18:30-19:30) |

**Please read the following instructions carefully:** Use of a basic calculator is allowed. Please provide an argument or calculation at every question. When you are finished, make photos of your solutions and upload them as **a single pdf file** to Canvas. Make sure the photo is clearly readable. Save your original solutions until your grade is determined.

**Please provide an argument or calculation at every question!**

**Question 1** [8 pnt]. Let

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}.$$

a) [3 pnt] Determine a basis for the null space of $A$.

**Solution:** A basis for the null space of $A$ can be obtained by solving $Ax = 0$ and writing the solution in parametric vector form:

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that the solution has the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} x_3,$$

hence a basis for the null space of $A$ is given by

$$\{ \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \}.$$

**Grading:** 1 pnt for recognizing that $Ax = 0$ needs to be solved, 1 pnt for bringing $A$ in reduced echelon form (and only 0.5 pnt if there is a mistake in the calculation), and 1 pnt for the correct answer.

b) [3 pnt] Determine a basis for the column space of $A$.

**Solution:** A basis for the column space of $A$ is given by the columns of $A$ that correspond to pivot columns. From the (reduced) echelon form in a) we see that the first two columns are pivot columns, so a basis for the column space of $A$ is given by

$$\{ \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \}.$$

**Grading:** 1 pnt for noting that a basis for the column space of $A$ is the set of pivot columns of $A$, 1 pnt for identifying the pivot columns, and 1 pnt for the correct answer.

The matrix $A$ can be viewed as a linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$.

c) [1 pnt] Is this transformation one-to-one? *Hint:* Use your answer to a).

**Solution:** No: One-to-one is equivalent to having a trivial null space (i.e. containing only the zero vector) but $A$ has a one-dimensional null space (see a)).

**Grading:** 0.5 pnt for the correct answer and 0.5 pnt for a correct argument.

d) [1 pnt] Is this transformation onto $\mathbb{R}^3$? *Hint:* Use your answer to b).

**Solution:** No: onto $\mathbb{R}^3$ is equivalent to having a three-dimensional column space, but $A$ has a two-dimensional column space (see b)).

**Grading:** 0.5 pnt for the correct answer and 0.5 pnt for a correct argument.

**Question 2** [6 pnt] Let $u = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $B = uu^T$.

a) [2 pnt] Compute $Bx$ and show that it equals the orthogonal projection of $x$ onto $u$.

**Solution:**

$$Bx = uu^T x = \frac{1}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \times 9 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},$$

and

$$\text{proj}_u(x) = \frac{\langle x, u\rangle}{||u||^2} u = \langle x, u\rangle u = 3 \times \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

**Grading:** 1 pnt for each answer (subtract 0.5 pnt for small mistake).

b) [2 pnt] Show that $B$ is symmetric and that $B^2 = B$.

**Solution:** $B^T = (uu^T)^T = (u^T)^T u^T = uu^T = B$ so $B$ is symmetric and $B^2 = (uu^T)(uu^T) = ||u||^2 uu^T = uu^T = B$. Comment: I suspect that the students will use the specific numerical vector $u$ (and that's fine; it's just a lot more work).

**Grading:** 1 pnt for each answer (subtract 0.5 pnt for small mistake).

c) [2 pnt] Show that $u$ is an eigenvector of $B$ and determine its eigenvalue.

**Solution:** $Bu = (uu^T)u = ||u||^2 u = 1 \times u$ so $u$ is an eigenvector of $B$ with eigenvalue 1.

**Grading:** 1.5 pnt for showing that $u$ is an eigenvector of $B$ (subtract 0.5 for each small mistake) and 0.5 pnt for determining its eigenvalue. Comment: I suspect that the students will use the specific numerical vector $u$ (and that's fine; ot's just a lot more work).

**Question 3** [5 pnt] Determine if the following statements are true or false. Provide an argument for your answer.

a) [1 pnt] If the system $Ax = b$ is inconsistent, then $b$ is not in the set spanned by the columns of $A$.

**Solution:** True: If $b$ would be in the column space of $A$, then $b$ could be written as a linear combination of the columns of $A$, in other words, there would be a coefficient vector $x$ such that $Ax = b$, but this system is inconsistent and hence such an $x$ cannot exist.

**Grading:** 0.5 pnt for correct answer and 0.5 pnt for a correct argument.

b) [1 pnt] An eigenspace of a matrix $A$ is the null space of a certain matrix.

**Solution:** True: The eigenspace of $A$ corresponding to eigenvalue $\lambda$ consists of the vectors $x$ that satisfy $(A-\lambda I)x = 0$, which is the null space of $A - \lambda I$.

**Grading:** 0.5 pnt for correct answer and 0.5 pnt for a correct argument.

c) [1 pnt] If $x$ is not in a subspace $W$, then $x - \text{proj}_W x$ is not zero.

**Solution:** True: There exist unique vectors $w \in W$ and $v \in W^\perp$ such that $x = w + v$. If $x$ is not in $W$, then $v \neq 0$ and hence $x - \text{proj}_W(x) = x - w = v \neq 0$.

**Grading:** 0.5 pnt for correct answer and 0.5 pnt for a correct argument.

d) [1 pnt] If $A$ is a $3 \times 2$ matrix, then the transformation $x \mapsto Ax$ cannot be onto $\mathbb{R}^3$.

**Solution:** True: $A$ has only two columns so the dimension of Col $A$ is at most 2. Since range $A$ = Col $A$, range $A$ can never span $\mathbb{R}^3$.

**Grading:** 0.5 pnt for correct answer and 0.5 pnt for a correct argument.

e) [1 pnt] An $n \times n$ symmetric matrix has $n$ distinct real eigenvalues.

**Solution:** False: The identity matrix has only one eigenvalue, namely 1 (with multiplicity $n$).

**Grading:** 0.5 pnt for correct answer and 0.5 pnt for giving a counter-example.