Test 4, 2020

1a) Let $\underline{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\underline{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Use Gram-Schmidt:

$\underline{v}_1 = \underline{x}_1$

$$\underline{v}_2 = \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1}\underline{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

an orthogonal basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

1b) normalize the orthogonal basis:

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \quad \left(\text{or } Q = \begin{bmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{2}\sqrt{2} \\ \frac{1}{3}\sqrt{3} & 0 \\ -\frac{1}{3}\sqrt{3} & \frac{1}{2}\sqrt{2} \end{bmatrix} \right)$$

Then $$R = Q^T A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{3} & 3/\sqrt{3} \\ 0 & 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix}.$$

1c) By the Orthogonal Decomposition Theorem,

$\underline{b}_1 = \hat{\underline{b}} = \text{proj}_{\text{Col} A}(\underline{b})$ and $\underline{b}_2 = \underline{b} - \hat{\underline{b}}$.

$$\underline{b}_1 = \frac{\underline{b} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1}\underline{v}_1 + \frac{\underline{b} \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2}\underline{v}_2 = \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{8}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$$

$$\underline{b}_2 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

2a) $\langle p, p \rangle = p(1)^2 + p(2)^2 + p(3)^2 = 0$ if and only if

$p(1) = 0$, $p(2) = 0$ and $p(3) = 0$

But $p \in \mathbb{P}_2$ ($p(t) = a_0 + a_1 t + a_2 t^2$), so p can at most have two roots (and not three). So this happens if and only if $p(t) = 0$ for all $t \in \mathbb{R}$.

2b) $\|p\| = \sqrt{\langle p, p \rangle}$

$\langle p, p \rangle = p(1)^2 + p(2)^2 + p(3)^2 = 2^2 + 1^2 + 0^2 = 5$

So $\|p\| = \sqrt{5}$.

2c) $\hat{q} = \dfrac{\langle q, p \rangle}{\langle p, p \rangle} p = \dfrac{2 \cdot 2 + 0 \cdot 1 + -6 \cdot 0}{5} (3-t) = \frac{4}{5}(3-t)$

3a) False Countrexample:

Let $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{z} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Then $\underline{x} \cdot \underline{y} = 0$, $\underline{y} \cdot \underline{z} = 0$, but $\underline{x} \cdot \underline{z} = 4 \neq 0$.

3b) True

$\{\underline{v}_1, \underline{v}_2\}$ is an orthonormal set, so

$\underline{v}_1 \cdot \underline{v}_2 = 0$ and $\|\underline{v}_1\| = 1 = \|\underline{v}_2\|$.

U is an orthogonal matrix, so

$(U\underline{v}_1) \cdot (U\underline{v}_2) = \underline{v}_1 \cdot \underline{v}_2 = 0$ and $\|U\underline{v}_1\| = \|\underline{v}_1\| = 1$

and $\|U\underline{v}_2\| = \|\underline{v}_2\| = 1$.

(see Theorem 7)

It follows that $\{U\underline{v}_1, U\underline{v}_2\}$ is an orthonormal set.