Test 3 2020

1a) $\begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & 1-\lambda & 2 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1-\lambda \\ 1 & 1 \end{vmatrix} = (1-\lambda)(\lambda^2-2\lambda+1-2)+2-1+\lambda$

$= \lambda^2-2\lambda-1-\lambda^3+2\lambda^2+\lambda+1+\lambda = -\lambda^3+3\lambda^2=0 \Rightarrow \lambda^2(3-\lambda)=0.$

$\lambda=0 \vee \lambda=3.$

1b) $\lambda=0$ has multiplicity 2. We have to check if the dimension of the corresponding ~~eigen~~ eigenspace is 2.

$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Only 1 free variable, so the dimension of the eigenspace is 1.

Therefore A is <u>not</u> diagonalizable.

→ (Or equivalently: the sum of the dimensions of the eigenspaces should be 3.)

2a) λ is an eigenvalue of B, so the equation $(B-\lambda I)\underline{x}=\underline{0}$ must have a nontrivial solution. Therefore, $B-\lambda I$ must have less than n pivots (there is a free variable), and since $B-\lambda I$ is square, this results in at least one zero row in echelon form.

2b) $\begin{vmatrix} 4-\lambda & -5 \\ 10 & 6-\lambda \end{vmatrix} = (4-\lambda)(6-\lambda)+50=0$

$\lambda^2-10\lambda+74=0$

$(\lambda-5)^2=-49 \Rightarrow \lambda=5\pm 7i.$

2c) $\begin{bmatrix} -1-7i & -5 \\ 0 & 0 \end{bmatrix}$ results in an eigenvector $\begin{bmatrix} 5 \\ -1-7i \end{bmatrix} = \underline{v}$

$\begin{bmatrix} 0 & 0 \\ 10 & 1-7i \end{bmatrix}$ results in an eigenvector $\begin{bmatrix} 1-7i \\ -10 \end{bmatrix} = \underline{u}$

They should belong to the same eigenspace, so check that they are multiples: $(1-7i)\begin{bmatrix} 5 \\ -1-7i \end{bmatrix} = \begin{bmatrix} 5(1-7i) \\ -1+49i^2 \end{bmatrix} =$

$\begin{bmatrix} 5(1-7i) \\ -50 \end{bmatrix} = 5\begin{bmatrix} 1-7i \\ -10 \end{bmatrix}$; yes: $\frac{1-7i}{5}\underline{v}=\underline{u}.$

2d) $\lambda_2 = \overline{\lambda_1}$ and we then know that $\underline{v}_1 = \overline{\underline{v}_2}$. So simply take the congugate of a vector in the eigenspace of λ_1: eigenspace for λ_2 = $\text{Span}\left\{ \begin{bmatrix} 5 \\ -1+7i \end{bmatrix} \right\}$.

3a) <u>False</u>. Only true if $\underline{v}_1$ and $\underline{v}_2$ are in the same eigenspace.

Suppose $A\underline{v}_1 = \lambda_1 \underline{v}_1$ and $A\underline{v}_2 = \lambda_2 \underline{v}_2$, $\lambda_1 \neq \lambda_2$.

Then $A(\underline{v}_1 + \underline{v}_2) = A\underline{v}_1 + A\underline{v}_2 = \lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 \neq \lambda(\underline{v}_1 + \underline{v}_2)$, for any $\lambda \in \mathbb{R}$.

or counterexample: $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, then $\lambda_1 = 1$ and $\lambda_2 = 2$.

$\underline{v}_1$: $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{v}_2$: $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{v}_1 + \underline{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then

$$A(\underline{v}_1 + \underline{v}_2) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ for any } \lambda \in \mathbb{R}.$$

3b) <u>True</u>. A and B are similar, so there exists an invertible matrix P such that $B = PAP^{-1}$. If A is diagonalizable, then there exists an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$. Combined this gives $B = P(QDQ^{-1})P^{-1} = (PQ)D(PQ)^{-1}$, where D is diagonal and PQ is invertible. So B is diagonalizable.

4a) $M\underline{v}_1 = \begin{bmatrix} 1{,}8 & -0{,}6 \\ 0{,}8 & 0{,}2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4{,}2 \\ 2{,}8 \end{bmatrix} = 1{,}4\begin{bmatrix} 3 \\ 2 \end{bmatrix}$; $\lambda_1 = 1{,}4$

$M\underline{v}_2 = \begin{bmatrix} 1{,}8 & -0{,}6 \\ 0{,}8 & 0{,}2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0{,}6 \\ 1{,}2 \end{bmatrix} = 0{,}6\begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $\lambda_2 = 0{,}6$.

4b) $x_k = c_1 \lambda_1^k \underline{v}_1 + c_2 \lambda_2^k \underline{v}_2$

Find c_1 and c_2: $x_0 = c_1\underline{v}_1 + c_2\underline{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. this gives:

$c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Solve $\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 2 & 2 & 2 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \end{array}\right]$.

$x_k = -\frac{1}{2}(1{,}4)^k\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{3}{2}(0{,}6)^k\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

as $k \to \infty$, $x_k \to -\frac{1}{2}(1{,}4)^k\begin{bmatrix} 3 \\ 2 \end{bmatrix} \to -\infty$.

(the system diverges)

4c) Now $x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $c_1 = 0$ and $c_2 = 1$.

$x_k = (0{,}6)^k\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

as $k \to \infty$, $x_k \to 0$. (the system converges).

Bonus