Linear Algebra 2020, Test 1.

1) $$\left[\begin{array}{cccc|c} 3 & 8 & -7 & -5 & 9 \\ 0 & 6 & -3 & -6 & 9 \\ -3 & -12 & 9 & 9 & -15 \end{array}\right] \sim \left[\begin{array}{cccc|c} 3 & 8 & -7 & -5 & 9 \\ 0 & 2 & -1 & -2 & 3 \\ 0 & -4 & 2 & 4 & -6 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 3 & 0 & -3 & 3 & -3 \\ 0 & 2 & -1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1/2 & -1 & 3/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \; 1 \; ; \quad \begin{cases} x_1 = x_3 - x_4 - 1 \\ x_2 = \frac{1}{2}x_3 + x_4 + 3/2 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$$

solution: $$\underline{x} = x_3 \begin{bmatrix} 1 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3/2 \\ 0 \\ 0 \end{bmatrix} . \; 1$$

$-\frac{1}{2}$ per small error.

2)
a) $$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -2 & -1 & 4 & 3 \\ 1 & 3 & k & -2 \\ 2 & 1 & -4 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 6 & 9 \\ 0 & 1 & k-1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & k-3 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right] \; \frac{1}{2}$$

Inconsistent if $k = 3$. $\frac{1}{2}$

b) only trivial solution / no free variables $\frac{1}{2}$: $k \neq 3$. $\frac{1}{2}$

c) $$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ -1 & 4 & 3 \\ 3 & k & -2 \\ 1 & -4 & -3 \end{array}\right] \begin{matrix} +2r_2 \\ \\ +3r_2 \\ +r_2 \end{matrix} \sim \left[\begin{array}{cc|c} 0 & 9 & 9 \\ -1 & 4 & 3 \\ 0 & k+12 & 7 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -4 & -3 \\ 0 & 1 & 1 \\ 0 & k+12 & 7 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -4 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 7-(k+12) \\ 0 & 0 & 0 \end{array}\right] \; \frac{1}{2}$$

consistent if $7-(k+12) = 0$, so if $k = -5$. $\frac{1}{2}$ Then $\underline{y} \in \text{Span}\{\underline{v}_2, \underline{v}_3\}$

d) $\underline{v}_1, \underline{v}_2, \underline{v}_3$ span $\mathbb{R}^4$ if and only if there is a pivot in every row. $\frac{1}{2}$ This is not possible (more rows than columns). There is no $k \in \mathbb{R}$ such that $\text{Span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\} = \mathbb{R}^4$. $\frac{1}{2}$

Or: We need at least 4 vectors to span $\mathbb{R}^4$, so there is no $k \in \mathbb{R}$ such that $\text{Span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\} = \mathbb{R}^4$.

3a) <u>False</u> Counterexample: matrix with more rows than columns: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Let $\underline{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\underline{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. 1

Then $A\underline{x} = \underline{b}$ has a unique solution, but $A\underline{x} = \underline{c}$ is inconsistent.

(also OK if they <u>explain</u> that $A\underline{x} = \underline{c}$ can be inconsistent.)

3b) <u>True</u> More columns than rows, so there cannot be a pivot in every column. $\frac{1}{2}$
So there are free variables, which implies $A\underline{x} = \underline{0}$ has a nontrivial solution. $\frac{1}{2}$
Therefore the columns are linearly dependent.

3c) <u>False</u> $T(\underline{0}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq \underline{0}$. 1

– or <u>show</u> $T(c\underline{x}) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) \neq cT\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

– or <u>show</u> $T(\underline{x} + \underline{y}) \neq T(\underline{x}) + T(\underline{y})$.

$\frac{1}{2}$ for only the definition: $T(c\underline{x}) = cT(\underline{x})$
$T(\underline{x} + \underline{y}) = T(\underline{x}) + T(\underline{y})$.