Vrije Universiteit Amsterdam, Faculty of Science **March 27, 2019**
Linear Algebra partial exam 1 **15.15-17.15**
grade = points / 40 ∗ 9 + 1

Use of calculators, books or notes is not allowed. Motivate your answers.

Question 1 (2p,2p,2p)

Given are the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 6 & p \\ -4 & -12 & 2 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 0 \\ 4 \\ q \end{bmatrix}$.

(a) Let $p = 4$ and $q = 12$. Solve the system $A\mathbf{x} = \mathbf{b}$.
(b) Determine the value(s) of p and q such that the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.
(c) Determine the value(s) of p and q such that the vector $\mathbf{b}$ is contained in the span of the columns of A.

Question 2 (2p,2p,1p)

Given is the matrix $B = \begin{bmatrix} 3 & -1 & 2 & 3 \\ -3 & 1 & 0 & -5 \\ 6 & -2 & 5 & 0 \end{bmatrix}$.

(a) Give the LU-factorization of the matrix B.
(b) Find an explicit description of the null space of the matrix B (i.e. Nul B).
(c) Find a basis for the column space of the matrix B (i.e. Col B).

Question 3 (2p,2p,3p)

Given are the matrix $C = \begin{bmatrix} -1 & 9 & 3 \\ 0 & x & 1 \\ x & -6 & 0 \end{bmatrix}$ with $x \in \mathbb{R}$ and the vector $\mathbf{d} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

(a) Compute the determinant of the matrix C in terms of x.
(b) For what value(s) of x is the matrix C invertible?
(c) Let $x = 3$. Determine using Cramer's rule the component y_2 of the system $C\mathbf{y} = \mathbf{d}$.

-EXAM CONTINUES ON THE BACK-

Question 4 (2p,3p,2p)

Let the transformation $T : \mathbb{P}_2 \to \mathbb{R}^3$ be defined by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(-1) - \mathbf{p}(0) \end{bmatrix}$,

where $\mathbb{P}_2 = \{\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ is the set of polynomials of degree at most 2.

(a) Prove that T is a linear transformation.
(b) Find a polynomial $\mathbf{p}$ in $\mathbb{P}_2$ that spans the kernel of T.
(c) Describe the range of T as a span of vectors.

Question 5 (2p,2p,2p,2p)

Mark each of the following two statements *true* or *false*. If the statement is *true*, give a proof. If the statement is *false*, give a proof or provide a counterexample.

(a) Let $\mathcal{B}$ be a basis for a vector space V. If the subset $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ of V is linearly independent, then the set of coordinate vectors $\{[\mathbf{v_1}]_\mathcal{B}, [\mathbf{v_2}]_\mathcal{B}, [\mathbf{v_3}]_\mathcal{B}\}$ is linearly independent.

(b) The set of vectors in and on the unit circle, given by $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \le 1 \right\}$, is a vector space.

(c) Let A and B be $n \times n$ matrices. If AB is invertible, then both A and B are invertible.

(d) Let A be an $m \times n$ matrix. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a one-to-one linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the rank of A is equal to n.

Question 6 (2p,3p,2p)

Let $H = \left\{ \begin{bmatrix} a+2b \\ a \\ 3b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ be the set of all vectors of the form $\begin{bmatrix} a+2b \\ a \\ 3b \end{bmatrix}$.

(a) Prove that H is a subspace of $\mathbb{R}^3$.

Given are bases $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right\}$ of H.

(b) Show that the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ is given by $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$.
(c) Find the change-of-coordinates matrix $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$ from $\mathcal{C}$ to $\mathcal{B}$.

-END OF EXAM-