Linear algebra Exam 1, 27/3/2019

1a) $\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 2 & 6 & 4 & 4 \\ -4 & -12 & 2 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 6 & 5 & 4 \\ 0 & -12 & 0 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 0 & -1 \end{array}\right] \sim$

$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ $\left\{\begin{array}{l} x_1 = 1 \\ x_2 = -1 \\ x_3 = 2 \end{array}\right.$

or $\underline{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

1b) $\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 2 & 6 & p & 4 \\ -4 & -12 & 2 & q \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 2 & 6 & p & 4 \\ 0 & 0 & 2+2p & q+8 \end{array}\right].$

Inconsistent if $2+2p=0$ and $q+8 \neq 0$

So if $p=-1$ and $q \neq -8$.

1c) That means $A\underline{x} = \underline{b}$ is consistent:

If $2+2p \neq 0$, so if $p \neq -1$.

or if $2+2p=0$ and $q+8=0$, so if $p=-1$ and $q=-8$.

2a) $B = \begin{bmatrix} 3 & -1 & 2 & 3 \\ -3 & 1 & 0 & -5 \\ 6 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 & 3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 & 3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

$= U.$

$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1/2 & 1 \end{bmatrix}.$

2b) $[B \mid \underline{0}] \sim [U \mid \underline{0}] \sim \left[\begin{array}{cccc|c} 3 & -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim$

$\left[\begin{array}{cccc|c} 1 & -1/3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$ $\left\{\begin{array}{l} x_1 = 1/3\, x_2 \\ x_2 \text{ is free} \\ x_3 = 0 \\ x_4 = 0 \end{array}\right.$, $\underline{x} = x_2 \begin{bmatrix} 1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, so

$\text{Nul } B = \text{Span}\left\{ \begin{bmatrix} 1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$

2c) Basis Col B is formed by the pivot columns of B; columns 1, 3 and 4.

$$\left\{ \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} \right\}$$

3a) $$\begin{vmatrix} -1 & 9 & 3 \\ 0 & x & 1 \\ x & -6 & 0 \end{vmatrix} = x \begin{vmatrix} -1 & 3 \\ x & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & 9 \\ x & -6 \end{vmatrix}$$

$$= x(0 - 3x) - 1 \cdot (6 - 9x)$$

$$= -3x^2 - 6 + 9x.$$

3b) Invertible if $\det C \neq 0$.

$$\det C = -3x^2 - 6 + 9x = 0$$

$$x^2 - 3x + 2 = 0$$

$$(x-1)(x-2) = 0$$

$$x = 1 \quad \vee \quad x = 2.$$

Invertible if $x \neq 1$ and $x \neq 2$.

3c) $x = 3$ then $\det C = -3(3)^2 - 6 + 9(3)$

$$= -27 - 6 + 27 = -6.$$

Cramer's Rule: $y_2 = \dfrac{\det C_2(\underline{d})}{\det C}$.

$$\det C_2(\underline{d}) = \begin{vmatrix} -1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 2 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= -1(0 - 2) + 3(2 - 6)$$

$$= 2 - 12 = -10.$$

$$y_2 = \frac{-10}{-6} = \frac{5}{3}.$$

4a) Let $p, q \in \mathbb{P}_2$ and $c \in \mathbb{R}$.

$$T(p+q) = \begin{bmatrix} (p+q)(-1) \\ (p+q)(0) \\ (p+q)(-1) - (p+q)(0) \end{bmatrix} = \begin{bmatrix} p(-1) + q(-1) \\ p(0) + q(0) \\ p(-1) + q(-1) - p(0) - q(0) \end{bmatrix}$$

$$= \begin{bmatrix} p(-1) \\ p(0) \\ p(-1) - p(0) \end{bmatrix} + \begin{bmatrix} q(-1) \\ q(0) \\ q(-1) - q(0) \end{bmatrix} = T(p) + T(q).$$

$$T(cp) = \begin{bmatrix} (cp)(-1) \\ (cp)(0) \\ (cp)(-1) - (cp)(0) \end{bmatrix} = \begin{bmatrix} cp(-1) \\ cp(0) \\ cp(-1) - cp(0) \end{bmatrix}$$

$$= c \begin{bmatrix} p(-1) \\ p(0) \\ p(-1) - p(0) \end{bmatrix} = cT(p).$$

4b) $T(p) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So $p(-1) = 0$, $p(0) = 0$ and $p(-1) - p(0) = 0$.

$p(t) = a_0 + a_1 t + a_2 t^2$.

$$T(p) = \begin{bmatrix} a_0 - a_1 + a_2 \\ a_0 \\ -a_1 + a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

$\begin{cases} a_0 = 0 \\ a_1 = a_2 \\ a_2 \text{ is free} \end{cases}$, $a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, so $p(t) = t + t^2$ spans the kernel of T.

4c) Range is $T(p) = \begin{bmatrix} a_0 - a_1 + a_2 \\ a_0 \\ -a_1 + a_2 \end{bmatrix} = a_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

So range $T = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

5a) True. The equation $c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 = \underline{0}$ has only the trivial solution.

It follows that $[c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3]_{\mathcal{B}} = [\underline{0}]_{\mathcal{B}}$ has the same solution c_1, c_2, c_3 because the coordinate mapping is one-to-one.

Because the coordinate mapping is linear,

$$c_1 [\underline{v}_1]_{\mathcal{B}} + c_2 [\underline{v}_2]_{\mathcal{B}} + c_3 [\underline{v}_3]_{\mathcal{B}} = \underline{0}.$$

It follows that this equation only has the trivial solution, so $\{[\underline{v}_1]_{\mathcal{B}}, [\underline{v}_2]_{\mathcal{B}}, [\underline{v}_3]_{\mathcal{B}}\}$ is linearly independent.

(Or prove that if $\{[\underline{v}_1]_{\mathcal{B}}, [\underline{v}_2]_{\mathcal{B}}, [\underline{v}_3]_{\mathcal{B}}\}$ is dependent, then $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is dependent.)

5b) False. Not closed under addition or scalar multiplication.

Counterexample: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in U$ but $2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin U$.

5c) True. $\det AB \neq 0$.

$\det AB = \det A \cdot \det B \neq 0$.

so $\det A \neq 0$ and $\det B \neq 0$.

Therefore A and B are invertible (because both $n \times n$).

5d) True. A has n columns. T is one-to-one, so A has a pivot in every column.

So the rank of A is n.

6a) $\begin{bmatrix} a+2b \\ a \\ 3b \end{bmatrix} \in \mathbb{R}^3$

$= a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}2\\0\\3\end{bmatrix}$, so $H = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\3\end{bmatrix}\right\}$.

Therefore H is a subspace of $\mathbb{R}^3$.

Or: let $\underline{u}, \underline{v} \in H$, then $\underline{u} = \begin{bmatrix} a+2b \\ a \\ 3b \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} x+2y \\ x \\ 3y \end{bmatrix}$.

Then $\underline{u}+\underline{v} = \begin{bmatrix} a+2b+x+2y \\ a+x \\ 3b+3y \end{bmatrix} = \begin{bmatrix} (a+x)+2(b+y) \\ a+x \\ 3(b+y) \end{bmatrix} \in H$

let $c \in \mathbb{R}$, then $c\underline{u} = \begin{bmatrix} c(a+2b) \\ ca \\ c \cdot 3b \end{bmatrix} = \begin{bmatrix} ca+2cb \\ ca \\ 3cb \end{bmatrix} \in H$.

This proves H is a subspace of $\mathbb{R}^3$.

6b) $\underset{C \leftarrow B}{P} = \begin{bmatrix} [\underline{b}_1]_C & [\underline{b}_2]_C \end{bmatrix}$

$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 5 \\ 1 & 0 & 2 & 3 \\ 0 & 3 & -3 & 3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 0 & 2 & -2 & 2 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So $\underset{C \leftarrow B}{P} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$.

6c) $\underset{B \leftarrow C}{P} = \left(\underset{C \leftarrow B}{P}\right)^{-1}$

$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 0 & 5 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} -1 & 0 & -1/5 & 3/5 \\ 0 & 1 & 1/5 & 2/5 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1/5 & -3/5 \\ 0 & 1 & 1/5 & 2/5 \end{array}\right]$ $\quad \underset{B \leftarrow C}{P} = \begin{bmatrix} 1/5 & -3/5 \\ 1/5 & 2/5 \end{bmatrix}$.