Faculty of Science
Department Mathematics, Vrije Universiteit

Final exam Linear Algebra 1BA
28-5-2018, 08:45–11:30

Use of calculator, book or notes is not allowed.

There are 6 exercises in this exam. Points are divided as follows.

1a:2	2a:3	3a2:	4a:2	5a:3	6a:2
b:1	b:2	b:4	b:1	b:3	b:2
c:1	c:1		c:1		c:2
			d:2		
			e:2		

The grade for this final exam will be computed as follows: Grade $= \frac{\#\text{points}}{4} + 1$

1. The matrix A is given by

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 5 & 5 \\ 2 & 2 & -1 \end{bmatrix}.$$

a. Determine a basis for $\operatorname{Nul} A$ as well as a basis for $\operatorname{Col} A$.
b. What is rank$A$?
c. Find an orthogonal basis for $\operatorname{Col} A$.

2. Let the matrix A be given by

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ p & 1 & 3 \end{bmatrix},$$

where p is a real number.
a. For which value of p does the subspace $\operatorname{Nul}(A - 2I)$ have dimension 2?
From here on p has the value you found in part **a.**
b. Find the eigenvalues and the corresponding eigenvectors of A.
c. Is A diagonalizable? If so why, if not why not?

3. Given are the vectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.
Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.
a. Why is the projection onto W of the vector $\mathbf{x}_4$ equal to the zero vector $\mathbf{0}$?
Let A be the 4×3 matrix whose columns are the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in that order.
b. Let $\mathbf{b}$ be the vector $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. Find the least squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$.
Also determine the least squares error.

4. Let $\mathbf{u} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\mathbf{x} = 2\mathbf{x} + 3(\mathbf{x} \bullet \mathbf{u}) \cdot \mathbf{u}.$$

a. Find the standard matrix A of T.
b. Compute $T\mathbf{u}$.
c. For $\mathbf{x} \perp \mathbf{u}$ compute $T\mathbf{x}$.
d. Find an orthonormal basis for $\text{Nul}\,(A - 2I)$.
e. Give a spectral decomposition of A.

5. Which of the following statements are true and which are false? Explain your answer when the statement is true, in case it is false give a counterexample.
a. If the columns of A are dependent then zero is an eigenvalue of A.
b. If $Q(\mathbf{x})$ is a quadratic form then there is a unique matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

6. Let A be a symmetric matrix and let $A = U\Sigma V^T$ be the singular value decomposition of A, and $A = WDW^T$ the spectral decomposition of A. The goal of this exercise is to study a possible relation between the two.
a. Take $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. What are the eigenvalues of A and the eigenvectors of A?
b. For A as in part **a.** what are the singular values? How would you be able to find U and V and Σ for this example in terms of the eigenvalues and eigenvectors of A?
c. Describe how in general the singular value decomposition for a symmetric matrix A is related to the spectral decomposition.