Linear algebra, exam 2, May 2018

1a) $A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 5 & 5 \\ 2 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$

$\begin{cases} x_1 = \frac{5}{2} x_3 \\ x_2 = -2x_3 \\ x_3 = x_3 \text{ is free} \end{cases} \rightarrow \underline{x} = x_3 \begin{bmatrix} 5/2 \\ -2 \\ 1 \end{bmatrix}$ a basis for Nul A is $\left\{ \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix} \right\}$.

Pivots in columns 1 and 2, so a basis for Col A is $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \right\}$.

1b) 2 pivots, so Rank A = dim Col A = 2.

1c) Use Gram-Schmidt:

$\underline{v}_1 = \underline{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$\underline{v}_2 = \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} - \frac{18}{12} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$

Orthogonal basis for Col A is $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$.

2a) $A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ p & 1 & 1 \end{bmatrix}$. If $p = -1$.

2b) $\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 = x_2 + x_3 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases} \rightarrow \underline{x} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

So $\lambda = 2$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Find other eigenvalue:

$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(3-\lambda) - (2-\lambda) = 0$

$(2-\lambda)\left((3-\lambda)(3-\lambda) - 1 \right) = 0$

$\lambda = 2 \quad \vee \quad \lambda^2 - 6\lambda + 8 = 0$

$(\lambda - 2)(\lambda - 4) = 0$

$\lambda = 2 \vee \lambda = 4.$

$\lambda = 4: \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 = 0 \\ x_2 = x_3 \\ x_3 \text{ free} \end{cases} \rightarrow \underline{x} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$

So eigenvector for $\lambda = 4$ is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

2c) Yes, because the dimension of each eigenspace equals the multiplicity of the corresponding eigenvalue.

3a) $\underline{x}_4$ is orthogonal to $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$, and the orthogonal projection is

$$\hat{\underline{x}}_4 = \frac{\underline{x}_4\cdot\underline{x}_1}{\underline{x}_1\cdot\underline{x}_1}\underline{x}_1 + \frac{\underline{x}_4\cdot\underline{x}_2}{\underline{x}_2\cdot\underline{x}_2}\underline{x}_2 + \frac{\underline{x}_4\cdot\underline{x}_3}{\underline{x}_3\cdot\underline{x}_3}\underline{x}_3 = 0.$$

$\underline{x}_1\cdot\underline{x}_4 = 0$, $\underline{x}_2\cdot\underline{x}_4 = 0$, $\underline{x}_3\cdot\underline{x}_4 = 0$.

3b) Solve the normal equations: $A^T A\hat{\underline{x}} = A^T \underline{b}$

$$A^T A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^T\underline{b} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

Solve $\left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

So $\hat{\underline{x}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The least-squares error is

$$\|\underline{b} - A\hat{\underline{x}}\| = \left\|\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}\right\|$$

$$= \left\|\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\| = \sqrt{4} = 2.$$

4a) $$T(\underline{e}_1) = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\cdot\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \frac{3}{\sqrt{3}}\cdot\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

$$T(\underline{e}_2) = 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\cdot\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \frac{3}{\sqrt{3}}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$T(\underline{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, so $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ such that $T(\underline{x}) = A\underline{x}$.

4b) $$T(\underline{u}) = 2\cdot\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)\cdot\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{2}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\cdot\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

4c) $\underline{x} \perp \underline{u}$ means $\underline{x}$ and $\underline{u}$ are orthogonal, so $\underline{x}\cdot\underline{u} = 0$.

Then $T(\underline{x}) = 2\underline{x} + \underline{0} = 2\underline{x}$.

4d) $$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 = -x_2 - x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases} \rightarrow \underline{x} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So $\mathrm{Nul}(A - 2I) = \mathrm{Span}\left\{\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{not orthogonal}}\right\}$.

not orthogonal $\rightarrow$ Gram-Schmidt.

$\underline{v}_1 = x_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\underline{v}_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} \underline{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$. Use $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Normalize: $\|\underline{v}_1\| = \sqrt{2}$, $\|\underline{v}_2\| = \sqrt{6}$.

Orthonormal basis for $Nul(A - 2I)$ is $\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \right\}$.

4e) Spectral decomposition : $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T = PDP^T$.
= orthogonal diagonalization of A.

From (d) we know $\lambda = 2$ is an eigenvalue of A with eigenvectors $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$.

From (b) we know $\lambda = 5$ is an eigenvalue of A with eigenvector $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

So $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

such that $A = PDP^T$.

5a) This is false in general. It is only true if A is square.
Nonsquare matrices don't have eigenvalues.
If A is square, then it is true by the
Invertible Matrix Theorem.

5b) Not part of the material this year.

6a) $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$

$1 - \lambda = \pm 2$

$\lambda_1 = 3$ and $\lambda_2 = -1$ are the eigenvalues of A.

$\lambda_1 = 3: \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \to \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1: \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to \underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

the corresponding eigenvectors of A.

6b) The singular values of A are the square roots of the eigenvalues of $A^T A$.

$$A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 16 = 0$$

$$5 - \lambda = \pm 4$$

$$\lambda_1^* = +9, \quad \lambda_2^* = 1.$$

$$\sigma_1 = 3, \quad \sigma_2 = 1 \quad \leftarrow$$ singular values of A. compare with (a)!

$$\lambda_1 = 9: \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \to \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1: \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to \underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

} same as before! So we did not have to do these steps.

From (a) we know $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ (normalize the eigenvectors)

and for Σ use $|\lambda_1|$ and $|\lambda_2|$: $\Sigma = \begin{bmatrix} |\lambda_1| & 0 \\ 0 & |\lambda_2| \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

To compute U, let $\underline{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$$U = \begin{bmatrix} \frac{A\underline{v}_1}{|\lambda_1|} & \frac{A\underline{v}_2}{|\lambda_2|} \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1 \underline{v}_1}{|\lambda_1|} & \frac{\lambda_2 \underline{v}_2}{|\lambda_2|} \end{bmatrix} = \begin{bmatrix} \underline{v}_1 & -\underline{v}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \text{sign}(\lambda_1)\underline{v}_1 & \text{sign}(\lambda_2)\underline{v}_2 \end{bmatrix}$$

$$\Bigg(A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

To compare: $A = PDP^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \Bigg)$

6c) In general:

Let A be a symmetric matrix.

Then $A = U\Sigma V^T$ and $A = PDP^T$ with:

$V = P$: matrix with normalized, orthogonal eigenvectors of A.

If $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$ then $\Sigma = \begin{bmatrix} |\lambda_1| & 0 & \cdots & 0 \\ 0 & |\lambda_2| & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & |\lambda_n| \end{bmatrix}$.

and $U = \begin{bmatrix} \text{sign}(\lambda_1)\underline{v}_1 & \text{sign}(\lambda_2)\underline{v}_2 & \cdots & \text{sign}(\lambda_n)\underline{v}_n \end{bmatrix}$.

with $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ the orthogonal, normalized eigenvectors of A.