Use of calculator, book or notes is not allowed.

1. Let p be a real number. For every p let the matrix A be given by

$$A = \begin{bmatrix} 0 & p & -1 \\ p & -1 & 0 \\ -1 & 0 & p \end{bmatrix}.$$

a. For which value of p is A not invertible?
b. Determine for that value Null (A), the null space of A.
c. For that value of p determine for which vectors $\mathbf{b}$ the system of equations $A\mathbf{x} = \mathbf{b}$ is consistent.
d. Now let $p = -1$. Determine the inverse of A in this case.

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ -x_1 + x_2 - x_3 + x_4 \\ x_1 + x_3 \\ x_2 + x_4 \end{bmatrix}.$$

a. Determine the standard matrix A of the linear transformation T.
b. Is T one-to-one? If not determine a basis for the null space of T.
c. Is T onto? Determine a basis for the range of T.
d. What is the rank of A?

3 a. Compute the following determinants:

$$\begin{vmatrix} 1 & a_1 \\ a_1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & a_2 \\ 0 & 1 & a_1 \\ a_2 & a_1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_1 \\ a_3 & a_2 & a_1 & 1 \end{vmatrix}.$$

b. Use your answer of part **a.** to guess the determinant of

$$\begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 & a_n \\ 0 & 1 & 0 & \cdots & 0 & a_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & a_1 \\ a_n & a_{n-1} & \cdots & \cdots & a_1 & 1 \end{vmatrix},$$

and explain how this can be proved by row operations.

4. Mark each of the following statements true or false. Justify your answer: If true give an argument why it is true, if false give a counterexample. In each statement A is an $m \times n$ matrix.
a. If the columns of A are independent, then A is invertible.
b. The rank of A is the dimension of the column space of A.
c. If there is a vector $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent, then the number of columns of A is less than the number of rows of A.
d. If the number of columns of A is less than the number of rows of A, then there is a vector $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent.

5. In the space $\mathbb{P}_2$ of poynomials of degree 2 or less, consider the set $\mathcal{B}$ with the elements

$$\mathbf{p}_1(t) = 2t + 1, \mathbf{p}_2(t) = 3t - 2, \mathbf{p}_3(t) = t^2 - 1.$$

a. Show that $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent.
b. Explain why the set $\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for $\mathbb{P}_2$.
c. Find the coordinates of $\mathbf{p}(t) = 3t^2 + 2t + 5$ with respect to the basis $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.
d. Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation $T(\mathbf{p}) = \begin{bmatrix} p'(0) \\ p'(1) \end{bmatrix}$.
Find the matrix of T with respect to the basis $\mathcal{B}$ in $\mathbb{P}_2$ and the standard basis in $\mathbb{R}^2$.

Points

1a:2	2a:2	3a:3	4a:$1\frac{1}{2}$	5a:2
b:2	b:3	b:3	b:$1\frac{1}{2}$	b:1
c:2	c:2		c:$1\frac{1}{2}$	c:3
d:2	d:1		d:$1\frac{1}{2}$	d:2

$$\text{Grade} = \frac{\#\text{points}}{4} + 1$$