Partial exam 1, 2018.

1a) $A \sim \begin{bmatrix} 0 & p & -1 \\ 0 & -1 & p^2 \\ -1 & 0 & p \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & p^3-1 \\ 0 & -1 & p^2 \\ -1 & 0 & p \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & p \\ 0 & -1 & p^2 \\ 0 & 0 & p^3-1 \end{bmatrix}$

A not invertible if not 3 pivots, so if $p^3-1=0$,
[Or: compute $\det A = -p^3+1=0$.] so if $p=1$.

1b) $A \sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Solve $A\underline{x}=0$.

This gives $\begin{cases} x_1 = x_3 \\ x_2 = x_3 \\ x_3 \text{ is free} \end{cases}$, $\underline{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

So $\text{Nul } A = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

1c) $A\underline{x} = \underline{b}$. Solve $\left[\begin{array}{ccc|c} 0 & 1 & -1 & b_1 \\ 1 & -1 & 0 & b_2 \\ -1 & 0 & 1 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & -1 & b_1 \\ 1 & -1 & 0 & b_2 \\ 0 & -1 & 1 & b_2+b_3 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 0 & 1 & -1 & b_1 \\ 1 & -1 & 0 & b_2 \\ 0 & 0 & 0 & b_1+b_2+b_3 \end{array}\right]$. Consistent if $b_1+b_2+b_3=0$.

$A\underline{x}=\underline{b}$ is consistent for vectors $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ with $b_1+b_2+b_3=0$.

1d) $A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$

$\left[\begin{array}{ccc|ccc} 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ -1 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & -1/2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & -1/2 \end{array}\right]$

So $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \end{bmatrix}$.

2a) $T(\underline{x}) = x_1\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + x_2\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + x_3\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}1\\1\\0\\1\end{bmatrix}$

$= \begin{bmatrix}1&1&1&1\\-1&1&-1&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\underline{x}$. So $A = \begin{bmatrix}1&1&1&1\\-1&1&-1&1\\1&0&1&0\\0&1&0&1\end{bmatrix}$.

2b) $A \sim \begin{bmatrix}1&1&1&1\\0&2&0&2\\0&-1&0&-1\\0&1&0&1\end{bmatrix} \sim \begin{bmatrix}1&1&1&1\\0&1&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$

A does not have a pivot in every column, so T is not one-to-one.

Solve $A\underline{x} = \underline{0}$: $\begin{cases} x_1 = -x_3 \\ x_2 = -x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free}\end{cases}$, $\underline{x} = x_3\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + x_4\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$.

So $\text{Nul } A = \text{Span}\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right\}$.

Therefore basis for Nul A is $\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right\}$.

2c) A does not have a pivot in every row, so T is not onto.

The range of T is $T(\underline{x}) = x_1\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + x_2\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + x_3\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}1\\1\\0\\1\end{bmatrix}$

$= \text{Span}\left\{\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}\right\}$, contains <u>dependent</u> vectors.

See (2b): the matrix has a pivot in columns 1 and 2.

So basis range of T is $\left\{\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}\right\}$. (= basis Col A; range T = Col A)

2d) The rank of A is the number of pivots of A.

So rank A = 2.

3a) $\begin{vmatrix} 1 & a_1 \\ a_1 & 1 \end{vmatrix} = 1 - a_1^2$

$$\begin{vmatrix} 1 & 0 & a_2 \\ 0 & 1 & a_1 \\ a_2 & a_1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & a_2 \\ 0 & 1 & a_1 \\ 0 & a_1 & 1-a_2^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1-a_1^2-a_2^2 \end{vmatrix} = 1-a_1^2-a_2^2$$

$$\begin{vmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_1 \\ a_3 & a_2 & a_1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & a_2 & a_1 & 1-a_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & a_1 & 1-a_2^2-a_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1-a_1^2-a_2^2-a_3^2 \end{vmatrix} = 1-a_1^2-a_2^2-a_3^2.$$

3b) Guess: $1-a_1^2-a_2^2-\dots-a_n^2$.

The matrix is almost upper triangular. The determinant of a triangular matrix is equal to the product of the entries on the main diagonal. 1 point

To create an upper triangular matrix, we need to create zeros in the last row (except for the $(n+1)$th entry) using row replacement operations.

Row replacement operations do not affect the determinant of a matrix. 1 point

For $1 \le i \le n$, subtract the ith row a_{n-i+1} times from the last (the $(n+1)$th) row.

Then we get

1 point

$$\begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 & a_n \\ 0 & 1 & 0 & \cdots & 0 & a_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & a_1 \\ 0 & 0 & \cdots & \cdots & 0 & 1-a_1^2-a_2^2-\dots-a_n^2 \end{vmatrix}$$

$$= 1\cdot 1\cdot \ldots \cdot 1\cdot (1-a_1^2-a_2^2-\dots-a_n^2) = 1-a_1^2-a_2^2-\dots-a_n^2.$$

4a) False, A is $m \times n$ so if $m \neq n$ then A cannot be invertible.

counterexample: $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$. columns are linearly independent, but A is not invertible.

4b) True, the rank of A is equal to the number of pivot columns of A, and the pivot columns of A form a basis for the column space of A. The dimension of the column space of A is equal to the number of vectors in its basis.

4c) False. counterexample: $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

4d) True. There cannot be a pivot in every row, so there exists a vector $\underline{b}$ such that $A\underline{x} = \underline{b}$ does not have a solution. (this is Theorem 4 in chapter 1)

5a) Let $C = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$.

$[p_1]_C = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $[p_2]_C = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$, $[p_3]_C = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 7 & 2 \\ 0 & 0 & 1 \end{bmatrix} \left(\sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

↳ pivot in every column, so p_1, p_2, p_3 are linearly independent.

5b) p_1, p_2, p_3 also span $\mathbb{P}_2$ because matrix has pivot in every row. Therefore $\{p_1, p_2, p_3\}$ is a basis for $\mathbb{P}_2$.

5c) $$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 5 \\ 2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 5 \\ 0 & 7 & 2 & -8 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 8 \\ 0 & 7 & 0 & -14 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

so $[p]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$.

5d) w.r.t. basis $\mathcal{B}$: $p(t) = a_0 p_1 + a_1 p_2 + a_2 p_3 = a_0(2t+1) + a_1(3t-2) + a_2(t^2-1)$
$= 2a_0 t + a_0 + 3a_1 t - 2a_1 + a_2 t^2 - a_2$
$= (a_0 - 2a_1 - a_2) + (2a_0 + 3a_1)t + a_2 t^2$

$p'(t) = (2a_0 + 3a_1) + 2a_2 t$; $p'(0) = 2a_0 + 3a_1$ and $p'(1) = 2a_0 + 3a_1 + 2a_2$

$T(p) = \begin{bmatrix} 2a_0 + 3a_1 \\ 2a_0 + 3a_1 + 2a_2 \end{bmatrix} = a_0 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + a_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, so matrix is $\begin{bmatrix} 2 & 3 & 0 \\ 2 & 3 & 2 \end{bmatrix}$. check: $\cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.