Second Exam Linear Algebra

Faculteit der Exacte Wetenschappen, Vrije Universiteit
22 December 2017, 8:45 - 10:45.

The use of calculators or books is not permitted.
Motivate your answers.

Assignment 1
Let

$$A = \begin{bmatrix} -1 & 8 & 4 \\ 2 & -1 & -2 \\ -2 & 4 & 5 \end{bmatrix}.$$

a) Show that $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ is an eigenvector of A.

b) Determine the other eigenvalues of A and give a basis for *every* eigenspace of A.

c) Determine whether A is diagonalisable. If so, give the diagonalsiation, i.e., an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why A is not diagonalisable.

Assignment 2
Let

$$B = \begin{bmatrix} -1 & -5 & -4 \\ 3 & 1 & 4 \\ 1 & 3 & 8 \\ 1 & 1 & 0 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}.$$

a) Determine an *orthogonal* basis of $\operatorname{Col} B$.

b) Determine the orthogonal projection of $\mathbf{b}$ onto $\operatorname{Col} B$.

c) Determine all Least Squares solutions of the system $B\mathbf{x} = \mathbf{b}$.

Assignment 3
Let

$$C = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}.$$

a) Determine the orthogonal diagonalisation of $C^T C$.

b) Give a Singular Value Decomposition of C, i.e., a 3×3 orthogonal matrix U, a 2×2 orthogonal matrix V, and a 3×2 diagonal matrix Σ such that $C = U\Sigma V^T$.

Assignment 4

Determine whether the following statements are true or not true. If the statement is true, give a proof. If the statement is not true, give a proof or a counterexample.

a) Let $\mathbf{v}$ be a complex-valued eigenvector of a real-valued matrix A. Then $\bar{\mathbf{v}}$, the complex conjugate of $\mathbf{v}$, is also an eigenvector of A, corresponding to the same eigenvalue.

b) Let U and V be orthogonal matrices of the same dimension. Then UV^T is also an orthogonal matrix.

c) Let A be a symmetric matrix that defines a negative definite quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Then A is invertible.

d) Let V be a finite-dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. If $\mathbf{u}$ and $\mathbf{v}$ are two orthogonal vectors, then $||\mathbf{u} - \mathbf{v}|| = ||\mathbf{u} + \mathbf{v}||$.

Assignment 5

Consider the vector space $V = \mathbb{P}_2$ of all polynomials of degree less or equal to 2. Let the function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be given by

$$\langle \mathbf{u}(x), \mathbf{v}(x) \rangle = 2\mathbf{u}(-1)\mathbf{v}(-1) + \mathbf{u}(0)\mathbf{v}(0) + 2\mathbf{u}(1)\mathbf{v}(1).$$

Let $\mathcal{E} = \{1 + x, 1 + x^2, -1 + 2x + x^2\}$ be a basis for V. Let $T : V \to V$ be a linear transformation defined by

$$T(a_0 + a_1 x + a_2 x^2) = a_2 - a_1 + (a_0 + a_1)x + a_1 x^2.$$

a) Show that $\langle \cdot, \cdot \rangle$ is an inner product on V.

b) What is the distance between the polynomials $\mathbf{u}(x) = 1 + x^2$ and $\mathbf{v}(x) = 3 - x$ with respect to this inner product?

c) Determine $[T]_{\mathcal{E}}$, the matrix representation of T with respect to the basis $\mathcal{E}$.

Number of points

1:	2:	3:	4:	5:
a) 2	a) 4	a) 3	a) 3	a) 4
b) 4	b) 3	b) 3	b) 3	b) 2
c) 2	c) 3		c) 3	c) 3
			d) 3	
total: 8	10	6	12	9

$$\text{Final grade} = \frac{\#\text{ points}}{5} + 1.$$