First Exam Linear Algebra

Faculteit der Exacte Wetenschappen, Vrije Universiteit

24 October 2017, 8:45 - 10:45.

Use of calculators or books is not allowed. Motivate your answers.

Assignment 1

Let A and $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 4 & 5 & -5 \\ 3 & -6 & -6 & 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

a) Determine all solutions to the equation $A\mathbf{x} = \mathbf{b}$.

b) Give a basis of the null space of A.

c) Give a basis of the column space of A.

Assignment 2

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T(x, y) = \begin{bmatrix} x + 3y \\ -x + 2y \\ 2x - y \end{bmatrix}.$$

a) Give the standard matrix corresponding to this linear transformation.

b) Is this map surjective (“onto”)? Motivate your answer.

c) Is this map injective (“one-to-one”)? Motivate your answer.

Assignment 3

a) Show that for all $x \in \mathbb{R}$

$$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = (x-1)^2(x+2).$$

b) Let

$$B = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

Determine using Cramer’s Rule the element y_3 of the solution to the system $B\mathbf{y} = \mathbf{d}$. (Hint: you may use the result from 3a).)

Assignment 4
Determine the inverse of C, where

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 7 & 3 \\ 2 & 6 & 2 \end{bmatrix}.$$

Assignment 5
Determine whether the following statements are true or not. If the statement is true, give a proof. If it is not true, give a proof or provide a counter example.

a) For all matrices it is true that $(A - B)(A + B) = A^2 - B^2$.

b) Let $\mathbb{P}_n$ be the vector space of all polynomials of degree less or equal to n. If S a set of $n+1$ polynomials that spans $\mathbb{P}_n$, then S is a basis for $\mathbb{P}_n$.

c) Let A be an $m \times n$ matrix, and suppose that $m < n$. If A has a pivot position in each row, then the dimension of the column space of A is equal to n.

d) Let A be a 3×2 matrix, and $\mathbf{b}, \mathbf{c}$ both vectors from $\mathbb{R}^3$. Suppose $A\mathbf{x} = \mathbf{b}$ has a unique solution. Then $A\mathbf{x} = \mathbf{c}$ also has a unique solution.

e) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a surjective ("onto") linear transformation, then the dimension of the kernel of T is equal to $n - m$.

Assignment 6

a) Let V be the subset of $\mathbb{R}^3$ of all vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ for which $v_1 = -v_3$. Show that V is a subspace of $\mathbb{R}^3$.

b) Let the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by first rotating a vector $\mathbf{v}$ over 90 degrees counter-clockwise, and subsequently mirroring this vector in the line $x = y$. Determine the standard matrix that corresponds to this map.

Points per question

1: a) 3	2: a) 1	3: a) 4	4: a) 5	5: a) 3	6: a) 3
b) 2	b) 2	b) 3		b) 3	b) 3
c) 2	c) 2			c) 3	
				d) 3	
				e) 3	

$$\text{Final mark} = \frac{\#\text{ points}}{5} + 1.$$