Partial exam 1, 2017 October

1a) $\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & -1 \\ -2 & 4 & 5 & -5 & -2 \\ 3 & -6 & -6 & 8 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & -1 \\ 0 & 0 & 3 & 1 & -4 \\ 0 & 0 & -3 & -1 & 4 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 10/3 & -7/3 \\ 0 & 0 & 1 & 1/3 & -4/3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{cases} x_1 = 2x_2 - \frac{10}{3}x_4 - \frac{7}{3} \\ x_2 \text{ is free} \\ x_3 = -\frac{1}{3}x_4 - \frac{4}{3} \\ x_4 \text{ is free} \end{cases}$

$$\underline{x} = x_2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -\frac{10}{3} \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{7}{3} \\ 0 \\ -\frac{4}{3} \\ 0 \end{bmatrix}$$

1b) $A \sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 10/3 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ see (a): basis Nul A is $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10/3 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}\right\}$

1c) pivot columns are 1 and 3, so basis Col A is $\left\{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -6 \end{bmatrix}\right\}$.

2a) $T(x,y) = x\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ so $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}$.

(no explanation required)

2b) A has more rows than columns, so cannot have a pivot in every row. Therefore A is not onto.

2c) The two columns of A are not multiples, so are linearly independent. There is a pivot in every column, so A is one-to-one.

3a) $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = x \begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & x \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & x \\ 1 & 1 \end{vmatrix}$

$= x(x^2-1) - (x-1) + (1-x)$

$= x^3 - x - x + 1 + 1 - x$

$= x^3 - 3x + 2$

$= (x-1)^2(x+2)$

3b) $\det B = (3-1)^2(3+2) = 4 \cdot 5 = 20.$

Cramer's Rule : $y_3 = \dfrac{\det B_3(\underline{d})}{\det B}$

$|B_3(\underline{d})| = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 2 \\ 1 & 1 & 0 \end{vmatrix} = -2 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = -2(3-1) = -4$

So $y_3 = \dfrac{-4}{20} = -\dfrac{1}{5}.$

4) $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 4 & 7 & 3 & 0 & 1 & 0 \\ 2 & 6 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -4 & 1 & 0 \\ 0 & 2 & 0 & -2 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -5 & 1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -4 & 1 & 1/2 \\ 0 & 1 & 0 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & 5 & -1 & -1/2 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & -1/2 \\ 0 & 1 & 0 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & 5 & -1 & -1/2 \end{array}\right]$

5a) False, $(A-B)(A+B) = A^2 - BA + AB - B^2$.
Generally, it's not true that $BA = AB$.

5b) True. $\dim \mathbb{P}_n = n+1$, so S is a smallest possible set that spans $\mathbb{P}_n$. So S is a basis for $\mathbb{P}_n$.

5c) False. A has m pivots and $m<n$, so there are m pivot columns (and n columns), so $\dim \operatorname{Col} A = m \neq n$.

5d) False. $A: 3\times 2$, $A\underline{x} = \underline{b}$ unique solution, so no free variables, so A has 2 pivots.
counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

then $\underline{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ unique solution of $A\underline{x} = \underline{b}$.
But take $\underline{c} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, then $A\underline{x} = \underline{c}$ is inconsistent.

5e) True. T has standard matrix $A: m\times n$.
$\dim \operatorname{Ker} T = \dim \operatorname{Nul} A$
$\dim \operatorname{Col} A = m$ because T is (surjective) onto.
pivot in every row.
$\dim \operatorname{Col} A + \dim \operatorname{Nul} A = n$
$m + \dim \operatorname{Nul} A = n$.
so $\dim \operatorname{Ker} T = n - m$.

6a) $U = \left\{ \begin{bmatrix} -v_3 \\ v_2 \\ v_3 \end{bmatrix} : v_2, v_3 \in \mathbb{R} \right\}$.

easiest : $U = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

So U is a subspace.

or :

① $\underline{0} \in U$ because $\underline{0} = \begin{bmatrix} -0 \\ 0 \\ 0 \end{bmatrix}$.

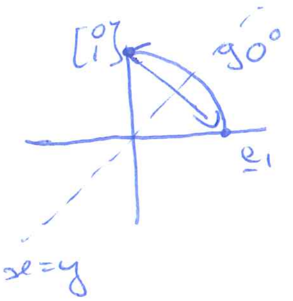
② Take $\underline{u}, \underline{v} \in U$, then $u = \begin{bmatrix} -u_3 \\ u_2 \\ u_3 \end{bmatrix}$ and $v = \begin{bmatrix} -v_3 \\ v_2 \\ v_3 \end{bmatrix}$

so $\underline{u} + \underline{v} = \begin{bmatrix} -u_3 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -v_3 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -u_3 - v_3 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} = \begin{bmatrix} -(u_3 + v_3) \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \in U$.

③ Take $\underline{u} \in U$ and $c \in \mathbb{R}$.

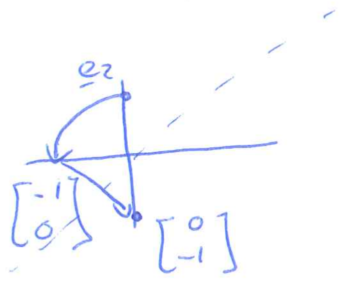
$c\underline{u} = c \begin{bmatrix} -u_3 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -cu_3 \\ cu_2 \\ cu_3 \end{bmatrix} \in U$.

6b) $A = [T(\underline{e}_1) \quad T(\underline{e}_2)]$. $\quad \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



$\underline{e}_1 \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underline{e}_1$

$T(\underline{e}_1) = \underline{e}_1$.



$\underline{e}_2 \to \begin{bmatrix} -1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$T(\underline{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

So $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.